



This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

- + *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.
- + *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.
- + *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.
- + *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at <http://books.google.com/>





Harvard College Library

FROM

Annals of Mathematics.

SCIENCE CENTER LIBRARY

INDEX

Sci 870.7
Q301 .1

TO THE

MATHEMATICAL GAZETTE

Vol. II.

JANUARY 1901, TO JANUARY 1904

(Nos. 25-43)

ONE SHILLING NET

LONDON ·

GEORGE BELL AND SONS

1904

Sci 870.94

1267222

THE Index is arranged according to subjects on the plan of the *Index du Répertoire Bibliographique* (published by MM. Gauthier-Villars, pp. 93, 2nd edition, 1898; about 2/-).

INDEX

TO THE

MATHEMATICAL GAZETTE

No. 25, JANUARY, 1901—No. 43, JANUARY, 1904.

1. Articles, etc.
 2. Notes.
 3. Reviews and Short Notices of Books.
 4. Solutions and Problems.
-

EXPLANATORY REMARKS.

It is not possible to give in full the subjects under the thousand or so subdivisions which appear in the *Index*; but for the sake of those who do not possess the *Index* we give the subjects under the principal letters. This will amply suffice for the majority of our readers.

- A. Algebra; Theory of Equations.
- B. Determinants; Linear substitutions; Elimination; Invariants and Covariants; Complex Quantities.
- C. Differential and Integral Calculus.
- D. Theory of Functions; Algebra of Continued Fractions.
- E. Definite Integrals. . . .
- F. Elliptic Functions. . . .
- G. Hyperelliptic Functions. . . .
- H. Differential Equations . . . Finite Differences, Recurrent Series.
- I. Arithmetic. Theory of Numbers.
- J. Permutations and Combinations; Probabilities; Calculus of Variations. . . .
- K. Elementary Geometry and Trigonometry; Geometry of point, line, plane, circle, and sphere; Perspective; Descriptive Geometry.
- L¹. Conics.
- L². Quadrics.
- M¹. Algebraical Plane Curves.
- M². Algebraical Surfaces.
- M³. Algebraical Gauche Curves.

- N. Complexes and Congruences; Connexes; Systems of curves and surfaces; Enumerative Geometry.
 O. Infinitesimal and Kinematic Geometry.
 P. Transformations; Homography; Homology; Polar Reciprocals.
 Q. Geometry of n dimensions; non-Euclidean. . . .
 R. Kinematics; Statics; Dynamics.
 S. Hydrostatics.
 T. Mathematical Physics.
 U. Astronomy.
 V. Philosophy and History of Mathematics. Biography.
 X. Graphic Calculation.

ARTICLES, ETC.

TYPE.	TITLE.	AUTHOR.	PAGE.
A. 1.	A Chapter on Algebra.	W. N. Roseveare.	301, 325
A. 1. b.	Proofs that the Arithmetic Mean is greater than the Geometric Mean.	R. F. Muirhead.	283
A. 3. k.	Note on the Solution of Cubic and Bi-quadratic Equations.	J. Brill.	3
B. 3. d.	Two Illustrations of Elimination.	R. F. Davis.	1
B. 12. a.	On the Representation of Imaginary Points by Real Points on a Plane.	A. Lodge.	277, 373
J. 4.	Some Applications of the Theory of Assemblages.	Arnold Emch.	173
K. 6. a.	Areal Coordinates.	G. D. Muggeridge.	45, 65
K. 6. a.	Notes on Conics and Areals.	T. J. I'a Bromwich.	175
K. 6. a.	Tripolar Coordinates.	G. N. Bates.	183
K. 6. a.	Tangential Coordinates, The use of.	R. W. H. T. Hudson.	355
K. 13. c.	The Trigonometry of the Tetrahedron.	G. Richardson.	149
K. 20.	Extension of Huygens' Approximation to a Circular Arc.	R. M. Milne.	309
L ¹ . 1.; L ² . 1.	On the treatment of Conics and Conicoids by Pure Geometry.	C. A. Rumsey.	356
M ¹ . 1. b.	Note on the Bitangents of a Plane Curve.	A. P. Thompson.	307
O. 1.	An Elementary Introduction to the Infinitesimal Geometry of Surfaces.	R. W. H. T. Hudson.	279
Q. d.	Geometry in Flatland.	A. C. Dixon.	241
R. 2. d.	On a Geometrico-Statistical Theorem.	Sir Robert S. Ball.	26
V. a.	The New Education Section of the British Association.	R. F. Muirhead.	83
V. a.	The Teaching of Mathematics.	E. M. Langley; C. Godfrey; A. W. Siddons.	106
V. a.	Annual Meeting, 1902. Discussion on Reform in Teaching of Mathematics.		129

INDEX.

v

TYPE.	TITLE.	AUTHOR.	PAGE.
V. a.	The Public Schools and Reform in the Teaching of Mathematics.	(By 22 Masters in Public Schools.)	143
V. a.	The Teaching of Euclid.	The Hon. Bertrand Russell.	165
V. a.	Report of Committee on Geometry.		168
V. a.	Report of Committee on Arithmetic and Algebra.		181
V. a.	Report of British Association Committee on the Teaching of Mathematics.		197
V. a.	The Report of the Committee on the Teaching of Geometry, and Incommensurables.	M. J. M. Hill.	253
V. a.	The Annual Meeting, 1903.		259
V. a.	The Committee on the Teaching of Mechanics.		349
V. a.	Draft Suggestions of the Subcommittee on the Teaching of Mechanics.		380
V. a.	Letter of the Committee to Head Masters and Senior Mathematical Masters of Schools.		349
V. a.	To reach the Calculus as early as possible.	G. H. Bryan.	351
X. 5.	The Slide Rule.	F. R. Barrell.	83; cf. 128
	" "	C. S. Jackson.	330; 337
X. 5.	Verniers, General Theory of.	P. J. Heawood.	221, 237

MATHEMATICAL NOTES.

	NO.	SUBJECT.	AUTHOR.	PAGE.
D. 2. b.	96	On the Convergency of the Geometric Series.	R. W. H. T. Hudson.	60
A. 1. b.	130	Note on Roseveare's "Chapter on Algebra."	R. F. Muirhead.	340
A. 1. c.	111	Note on the Multinomial Theorem.	G. Osborn.	189
C. 1. a.	106	On the Permutability of Independent Differentiations.	A. C. Dixon.	159
D. 1. b.	133	A Series Rapidly Converging.	G. N. Watson.	361
D. 2. a. β .	135	Riemann's semi-convergent series theorem, Proof of.	H. L. Trachtenburg.	361
D. 2. b.	124	Addition Series.	W. H. Laverly.	287
D. 2. d. a.	95	Periodic Continued Fractions.	T. Muir.	58
D. 6. b.	103	Note on the Logarithmic Series.	A. C. Dixon.	113
D. 6. d.; X. 2.	108	Two Minor Complaints.	C. S. Jackson.	188
D. 6. d.	109	Mnemonic for Hyperbolic Formulae.	G. Osborn.	189
D. 6. d.	114	On Complaints (cf. Note 108).	R. F. Muirhead.	201
D. 6. d.	114	" " "	R. B. Hayward.	201

	No.	SUBJECT.	AUTHOR.	PAGE.
E. 1. a.	127	Elementary Treatment for Gamma Functions.	G. Osborn.	311
I. 13. b. a; 17. c.	119	Rule for finding the number of quarts not greater than a given number N	Whitworth.	243
I. 25.	131	A Paradox.	C. S. Jackson.	360
J. 1. a.	115	A Problem in Arrangements.	Prebendary Whitworth and W. E. Hartley.	227
K.	93	On Measure as a Definition of Ratio.	E. Budden.	10, v. p. 30.
K. 1. b.	100	The Equal Bisector Theorem.	R. B. Hayward.	74
K. 2. b. c.	138	Feuerbach's Theorem.	W. F. Beard.	362
K. 4.	137	Philo's Line.	R. Chartres.	362
K. 6. a.	110	Length of join of $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$ in Trilinears.	Knox, Ma.	189
K. 6. a.	117	Trilinear Notes.	R. F. Davis.	225
K. 8. f.	136	The maximum quadrilateral with given sides.	R. Chartres.	362
K. 10. e.	94	The circum-circle of the triangle whose sides are $\Sigma a_1 x = 0$.	E. M. Radford.	29
K. 11. b.	102	The circles of similitude of the in-and-ex-circles of a triangle.	R. W. Chapman.	97
K. 11. b.	102	(Notes by R. F. Davis.)		99
K. 13. a.	92	Solution of Problem 372, and notes suggested thereby.	T. J. I'a Bromwich.	9
K. 17. b.	141	Some Simple Problems in Astronomy.	H. Hilton.	384
K. 20. d.	125	Approximation to $\tan A$.	C. E. M'Vicker.	238
K. 20. e.; 4.	104	On the construction of a triangle, given $\cos A : \cos B : \cos C$.	R. F. Davis.	159
K. 20. e.	112	Note on Problem 391.	F. G. Taylor.	191
K. 20. e.	126	On the Solution of Triangles.	G. H. Bryan.	238
K. 20. e.	129	Two Trigonometrical Notes.	R. F. Muirhead.	340
K. 20. f.	122	Formulae of Spherical Trigonometry.	C. S. Jackson.	261
L ¹ . 1. a.	123	Notes on the Parabola.	R. F. Davis.	262
L ¹ . 3. a.	118	Method of Reducing Central Conics.	E. L. Bulmer.	226
L ¹ . 3. a.	134	Note on Parabola.	R. F. Davis.	361
L ¹ . 6. c.	97	Trilinear Equation of the Circle of Curvature.	R. F. Davis.	72
L ¹ . 7. a.	101	Foci of in-Conic.	R. F. Davis.	97
L ¹ . 11.	98	Trilinear Conditions for Rectangular Hyperbola.	R. F. Davis.	73
L ¹ . 11. a.	113	The Asymptotes and Tangents of the Rectangular Hyperbola.	R. F. Davis.	192

INDEX.

vii

	No.	SUBJECT.	AUTHOR.	PAGE.
L. 12. a.	121	Isogonal Transformation.	H. L. Trachtenburg.	261
M. 8.	106	The Equiangular Spiral.	W. Gallatly.	159
P. 2. a.	99	Extension of Problem 394.	A. P. Thompson.	74
P. 3. b.	139	Geometrical Note on Inversion.	R. F. Davis.	383
R. 6. b.	128	Gauss' Principle of Least Constraint.	P. E. B. Jourdain.	337
R. 8.	132	Diagrams and Bending Moments.	C. S. Jackson.	360
V. 1. a.	120	On the teaching of Arithmetic.	G. H. Bryan.	260
V. 1. a.	126	On the Solution of Triangles.	G. H. Bryan.	288
V. 1. a.	140	On Decimalisation of Money.	W. O. Hemming.	383
X. 4. b. a.	116	The Graphical Solution of Quadratics.	A. E. Wynne.	228

[By an inadvertency, no Note was numbered 107.]

CORRESPONDENCE.

K.	On the Ratio of Incommensurables.	M. J. M. Hill.	30
V.	On the Marking of Euclid Papers.	C. S. Jackson.	8
V.	" " " "	A. W. Siddons.	58
V.	The Education Section of the British Association.	W. M. Heller.	371
V.	The Teaching of Geometry.	M. E. Boole.	180
V.	The London Mathematical Society.	G. H. Bryan.	396

REVIEWS AND NOTICES.

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
K.	Ahrens.	A First Book on Geometry.	E. M. Langley.	369
K.	Alasia, C.	Esercizi ed applicazioni di Trigonometria Piana.	W. J. G.	57
K.	Allcock.	Theoretical Geometry for Beginners. Part I.	E. M. Langley.	369
K.	Allcock.	Theoretical Geometry for Beginners. Part II.	E. M. Langley.	292
K.	Andrew.	Geometry.	E. M. Langley.	292
K.	Atkinson.	Practical Solid Geometry.	W. J. Dobbs.	210
I.	Bachmann.	Niedere Zahlentheorie.	W. J. G.	318
K.	Baker (and Bourne).	Elementary Geometry.	R. W. H. T. Hudson.	244
K.	"	" "	E. M. Langley.	292
U.	Ball.	A Primer of Astronomy.	W. J. G.	162
A.	Bardey.	Algebraische Gleichungen.	W. J. G.	315
K.	Barnard.	A New Geometry for Schools.	E. M. Langley.	369
K.	Barrell.	Elementary Geometry. Parts I. and II.	E. M. Langley.	217
K.	Barrell.	Elementary Geometry. Sect. II.	E. M. Langley.	369
A.	Bauer.	Vorlesungen über Algebra.	W. J. G.	391

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
V.	Beman.	v. Fink.		
C.	Bendt.	Katechismus der Differential- und Integralrechnung.	<i>W. J. G.</i>	218
A.	Bendt.	Katechismus der Algebra- ischen Analysis.	<i>W. J. G.</i>	219
A.	Bertrand (and Garoet).	Traité d'Algèbre.	<i>W. J. G.</i>	34
V.	Bertrand.	Éloges Académiques.	<i>W. J. G.</i>	317
K.	Blaikie and Thomson.	Geometrical Deductions. Books I. and II.	<i>W. J. G.</i>	218
X.	Blaine.	The Slide Rule.	<i>W. J. G.</i>	316
K.	Blythe.	Geometrical Drawing.	<i>W. J. Dobbs.</i>	53
K.	Böcher.	v. Briggs.		
D.	Borel.	Leçons sur les Séries Diver- gentes.	<i>E. T. Whittaker.</i>	92
K.	Bourne.	v. Baker.		
A; K.	Briggs.	Advanced Algebra and Geometry.	<i>E. M. Langley.</i>	293
K. L.	Briggs (and Böcher).	Elements of Plane Analytical Geometry.	<i>W. J. G.</i>	317
H.	Brown.	v. Burn.		
K.	Bryant (and Smith).	Euclid I.-IV., VI.-XI.	<i>W. J. G.</i>	116
K.	Burn, G. F.	First Stage Practical Plane and Solid Geometry.	<i>E. M. Langley.</i>	370
H.	Burn (and Brown).	Elements of Finite Differ- ences.	<i>W. J. G.</i>	317
A.	Burnside.	The Theory of Equations, with an Introduction to the Theory of Binary Algebraic Forms.	<i>E. B. Elliott.</i>	91
A.	Carrara.	Algebra.	<i>W. J. G.</i>	34
U.	Carrara.	La Selenografia, antica e moderna.	<i>W. J. G.</i>	162
V.	Carrara.	La Quadrature del Cerchio.	<i>W. J. G.</i>	294
V.	Castle.	Practical Mathematics for Beginners.	<i>W. J. G.</i>	147
K.	Child.	v. Barnard.		
R; X.	Charnock.	Exercises in Graphic Statics. Part I.	<i>W. J. Dobbs.</i>	53
R; X.	Charnock.	Exercises in Graphic Statics. Part II.	<i>W. J. Dobbs.</i>	211
I.	Chope.	Junior Arithmetic.	<i>W. J. G.</i>	395
B; I.	Combebiac.	Calcul de Triquaternions.	<i>C. J. Joly.</i>	202
U.	Comstock.	A Text-Book of Field As- tronomy for Beginners.	<i>C. J. Joly.</i>	267
V.	Cracknell.	Practical Mathematics.	<i>G. B. Mathews.</i>	52
K.	Croome-Smith.	Primer of Geometry.	<i>A. W. Siddons.</i>	212
R.	Cunningham.	First Stage Building Con- struction.	<i>W. J. Dobbs.</i>	211

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
V.	Curtze.	Urkunden zur Geschichte der Mathematik im Mittelalter und der Renaissance.	<i>W. J. G.</i>	214
J.	Czuber.	Probabilités et Moyennes Géométriques.	<i>W. J. G.</i>	245
J.	Czuber.	Wahrscheinlichkeitsrechnung. Part I.	<i>W. J. G.</i>	245
J.	Czuber.	Wahrscheinlichkeitsrechnung. Part II.	<i>W. R. Strong.</i>	346
V.	Dauzat.	Éléments de Méthodologie Mathématique.	<i>W. J. G.</i>	31
A.	David.	Beginners' Algebra.	<i>W. J. G.</i>	317
V.	Davison.	Easy Mathematical Problem Papers.	<i>W. J. G.</i>	230
A.	Deakin.	The Tutorial Algebra. Part I.	<i>W. J. G.</i>	147
K.	Deakin.	Euclid V., VI., XI.	<i>E. M. Langley.</i>	315
A.	Dickson.	Linear Groups, with an exposition of the Galois Field Theory.	<i>G. B. Mathews.</i>	204
A.	Dickson.	Introduction to the Theory of Algebraic Equations.	<i>E. B. Elliott.</i>	313
K.	Dilworth.	A New Sequel to Euclid.	<i>W. J. G.</i>	230
R.	Dobbs.	A Treatise on Elementary Statics.	<i>W. H. Macaulay.</i>	161
R.	Duncan.	Applied Mechanics for Beginners.	<i>A. Lodge.</i>	230
X.	Dunlop (and Jackson).	Slide Rule Notes. A Celluloid Slide Rule and Logologarithmic Slide.	<i>F. R. Burrell.</i> <i>W. J. G.</i>	205 317
K.	Duporcq.	Premiers Principes de Géométrie Moderne.	<i>W. J. G.</i>	116
R.	Durley.	Kinematics of Machines.	<i>W. J. G.</i>	392
K.	Edmondson.	Deductions in Euclid.	<i>W. J. G.</i>	116
C.	Edser.	Differential and Integral Calculus for Beginners.	<i>W. J. G.</i>	122
K.	Eggar.	Practical Exercises in Geometry.	<i>E. M. Langley.</i>	292
K.	Elliott.	Elementary Geometry.	<i>A. W. Siddons.</i>	245
R.	Emtage.	Elementary Mechanics of Solids.	<i>W. H. Macaulay.</i>	5
D. & c.	Estanave.	Sommatton de quelquesséries trigonométriques.	<i>W. J. G.</i>	394
K.	F. J.	Éléments de Géométrie.	<i>W. J. G.</i>	116
K.	Faudry.	Plane Geometrical Drawing.	<i>W. J. Dobbs.</i>	210
V.	Finkel.	A Mathematical Solution Book.	<i>R. F. Davis.</i>	342
K.	Finn.	The "Junior" Euclid. Books III. and IV.	<i>W. J. G.</i>	116

SUMMARY.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
V.	Fischer.	Der Naturwissenschaft Unterricht in England.	<i>W. J. G.</i>	218
K.	Fletcher.	Elementary Geometry.	<i>W. J. G.</i>	218
K.	Fletcher.	Elements of Geometry.	<i>E. M. Langley.</i>	292
D.	Fouet.	Leçons Élémentaires sur la théorie des Fonctions Analytiques.	<i>J. E. Wright.</i>	316
I.	Fourrey.	Récréations Arithmétiques.	<i>W. J. G.</i>	57, 294
C.	Forsyth.	Theory of Differential Equations. Part III.	<i>G. B. Mathews.</i>	204
C.	Forsyth.	A Treatise on Differential Equations.	<i>W. J. G.</i>	296
V.	Frankland.	The Story of Euclid.	<i>E. M. Langley.</i>	209
V.	Freycinet.	De l'Expérience en Géométrie.	<i>W. J. G.</i>	391
A.	Garost.	v. Bertrand.		
O.	Gauss.	General Investigations of Curved Surfaces, 1827 and 1825.	<i>W. J. G.</i>	215
V.	Geissler.	Die Grundsätze und das Wesen des Unendlichen.	<i>W. J. G.</i>	318
T.	Gleichen.	Lehrbuch der Geometrischen Optik.	<i>S. D. Chalmers.</i>	206
I.	Gmeiner.	Theoretische Arithmetik. Part I.	<i>G. B. Mathews.</i>	52
		Theoretische Arithmetik. Part II.	<i>G. H. Hardy.</i>	313
K.	Godfrey (and Siddons).	Elementary Geometry.	<i>E. M. Langley.</i>	369
D.	Godefroy.	La Fonction Gamma.	<i>T. J. Fa Bromwich.</i>	160
K.	Gournerie (de la).	Traité de Géométrie Descriptive.	<i>W. J. G.</i>	116
C.	Goursat.	Cours d'Analyse Mathématique. Tome I.	<i>J. E. Wright.</i>	244
C.	Graham.	An Elementary Treatise on the Calculus for Engineering Students.	<i>G. B. Mathews.</i>	52
A.	Graham.	Elementary Algebra.	<i>W. J. G.</i>	146
R.	Gray.	A Treatise on Physics. Vol. I.	<i>W. H. Macaulay.</i>	95
K. 22.	Gueth.	Projection Drawing.	<i>C. S. Jackson.</i>	390
K.	Guichard.	Traité de Géométrie.	<i>W. J. G.</i>	390
K.	Hadamard.	Géométrie dans l'Espace.	<i>W. J. G.</i>	116
K.	Hall (and Stevens).	A School Geometry. Parts I and II.	<i>E. M. Langley.</i>	315
K.	Hall (and Stevens).	A School Geometry. Part III.	<i>E. M. Langley.</i>	369
K.	Hamilton (and Kettle).	A First Geometry Book.	<i>W. J. Dobbs.</i>	53
I.	Hargreaves.	Arithmetic.	<i>E. M. Langley.</i>	206

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
K.	Harrison.	Practical Plane and Solid Geometry.	<i>E. M. Langley.</i>	316
R.	Henrici (and Turner).	Vectors and Rotors.	<i>W. J. G.</i>	393
D.	Hensel (and Landsberg).	Theorie der Algebraischen Funktionen einer Variablen, und ihre Anwendung auf Algebraische Kurven und Abelsche Integrale.	<i>G. B. Mathews.</i>	263
K ; U.	Hilbert.	Les Principes Fondamentaux de la Géométrie.	<i>W. J. G.</i>	116
K ; V.	Hilbert.	The Foundations of Geometry.	<i>W. J. G.</i>	268
L.	Hill.	A South African Arithmetic.	<i>E. M. Langley.</i>	206
R.	Hilton.	Mathematical Crystallography and the Theory of Groups of Movements.	<i>A. E. H. Tutton.</i>	387
K.	Hoevar.	Solid Geometry.	<i>E. M. Langley.</i>	316
K.	Holgate.	Elementary Geometry. Plane and Solid.	<i>W. J. G.</i>	116
R. ; S.	Hopkins.	Experimental Science.	<i>W. J. G.</i>	395
R.	Hoskins.	Theoretical Mechanics.	<i>W. H. Macaulay.</i>	162
V.	Ionescu.	Culegere de Probleme.	<i>W. J. G.</i>	217
X.	Jackson.	v. Dunlop.		
K.	Jocelyn.	High School and Academic Algebra.	<i>E. M. Langley.</i>	293
B ; I ; M.	Joly.	Quaternions and Projective Geometry.	<i>R. W. H. T. Hudson.</i>	370
K.	Jones.	The Beginnings of Trigonometry.	<i>W. J. G.</i>	219
L.	Jones, A. C.	Notes on Analytical Geometry.	<i>W. J. G.</i>	395
K.	Kettle.	v. Hamilton.		53
K.	Killing.	Die ebene Geometrie ; Die Geometrie des Raumes.	<i>T. J. Fa Bromwich.</i>	367
L.	Kitt.	Grundlinien der Politischen Arithmetik.	<i>W. J. G.</i>	217
V.	Koenigsberger.	Hermann von Helmholtz.	<i>W. J. G.</i>	394
	Knowles.	Calculating Scale.	<i>C. S. Jackson.</i>	389
F. I. a.	Kraser.	Lehrbuch der Theta-Funktionen.	<i>R. W. H. T. Hudson.</i>	342
K.	Lachlan.	Euclid. Book XI.	<i>A. W. Siddons.</i>	213
K.	Lachlan.	v. Fletcher.		
D.	Landsberg.	v. Hensel.		
R.	Lanza.	Applied Mechanics.	<i>T. J. Fa Bromwich.</i>	161
J.	Laplace.	A Philosophical Essay on Probabilities.	<i>W. J. G.</i>	245
K.	Larmor.	Solutions of Geometrical Exercises from Nixon's "Euclid Revised."	<i>R. F. Davis.</i>	115

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
B.	Laurent.	L'Elimination.	<i>T. J. Pa Bromwich.</i>	27
I.	Layng.	A Complete Short Course of Arithmetic.	<i>W. J. G.</i>	395
C.	Levy.	v. Rouché.		
C.	Linebarger (and Young).	The Elements of the Differential and Integral Calculus.	<i>G. B. Matheson.</i>	52
C.	Lodge.	Differential Calculus for Beginners.	<i>R. F. Davis.</i>	213
S.	Loney.	The Elements of Hydrostatics.	<i>W. J. G.</i>	123
M.	Loria.	Specielle Algebraische und Transcendente Ebene Kurven.	<i>W. J. G.</i>	267
L.	Lothian.	Arithmetic.	<i>E. M. Langley.</i>	206
X. 8.	Low.	Set Squares, Protractors and Scales.	<i>W. J. G.</i>	395
R.	Macgregor.	An Elementary Treatise on Kinematics and Dynamics.	<i>W. J. G.</i>	230
X. 2.	M'Aulay.	Five-figure Logarithmic and other Tables.	<i>W. J. G.</i>	394
R; V.	Mach.	The Science of Mechanics.	<i>W. J. G.</i>	230
R. S.	Maclean.	Exercises in Natural Philosophy.	<i>W. J. G.</i>	35
Q.	Manning.	Non-Euclidean Geometry.	<i>G. B. Halsted.</i>	94
C.	Mellor.	Higher Mathematics for Students of Chemistry and Physics.	<i>A. Lodge.</i>	223
C.	Meyer.	Differential- und Integralrechnung.	<i>W. J. G.</i>	250
K. L. ¹	Michel.	Recueil de Problèmes de Géométrie Analytique.	<i>W. J. G.</i>	33
A.	Milne.	Advanced Algebra.	<i>W. J. G.</i>	391
V.	Milner.	On Teaching Geometry.	<i>W. J. G.</i>	116
R.	Minchin.	The Student's Dynamics.	<i>W. H. Macaulay.</i>	5
X.	Morgan.	Elementary Graphs.	<i>W. J. G.</i>	393
A.	Mudie.	Factors in Algebra.	<i>E. M. Langley.</i>	370
K.	Mukerjee.	Elementary Geometry.	<i>E. M. Langley.</i>	315
A.	Mukerjee.	Elementary Algebra.	<i>E. M. Langley.</i>	370
V.	Müller.	Vocabulaire Mathématique. I. (Français-Allemand).	<i>W. J. G.</i>	56
V.	Müller.	Vocabulaire Mathématique. II. (Französisch-Deutsch).	<i>W. J. G.</i>	147
K. 20. f.	Murray.	Spherical Trigonometry.	<i>W. J. G.</i>	392
K.	Nesbitt.	Inductive Geometry for Transition Classes.	<i>W. J. Dobbs.</i>	53
J.	Netto.	Lehrbuch der Combinatorik.	<i>W. J. G.</i>	216
C.	Nichols.	Differential and Integral Calculus.	<i>W. J. G.</i>	122

INDEX.

xiii

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
V.	Ormsby.	Elementary Practical Mathematics.	W. J. G.	147
A.	Panton.	v. Burnside.		
K.	Paul.	v. Ahrens.		
K.	Pascal.	Repertorium der höheren Mathematik—Geometrie.	W. J. G.	219
J.	Pascal.	I Gruppi Continui di Trasformazioni.	E. B. Elliott.	264
K.	Petch.	Plane Geometry.	E. M. Langley.	292
K.	Plant.	Geometrical Drawing for Army and Navy Candidates, etc.	W. J. Dobbs.	53
L.	Puckle.	Conic Sections and Algebraic Geometry.	W. J. G.	390
R.	Pullen.	Mechanics, Theoretical Applied and Experimental.	C. S. Jackson.	346
R.	Roberts, H. A.	A Treatise on Elementary Dynamics.	W. H. Macaulay.	5
K.	Roberts, R.	A New Geometry for Beginners.	E. M. Langley.	292
K ; X.	Row (Sundara).	Geometric Exercises in Paper Folding.	E. M. Langley.	209
O.	Rouché.	Analyse Infinitésimale.	T. J. Fa Bromwich.	72
K ; L.	Rudio.	Die Analytische Geometrie des Raumes.	W. J. G.	218
V.	Rupert.	Famous Geometrical Theorems and Problems.	W. J. G.	116
K ; V.	Russell.	Essai sur les Fondements de la Géométrie.	W. J. G.	116
K.	Schults (and Sevenoak).	Plane and Solid Geometry.	W. J. G.	116
K.	Schuster.	Planimetrie, Stereometrie, Ebene und Sphärische Trigonometrie.	W. J. G.	390
K.	Sevenoak.	v. Schultze.		
K.	Siddons.	v. Godfrey.		
K.	Smith, C.	v. Bryant.		
K.	„ C.	„		
V.	„ D. E.	v. Fink.		
K.	Smith, C.	Solutions to Smith and Bryant's Geometry.	W. J. G.	219
K.	Smith.	v. Croome Smith.		
V.	Smith, D. E.	The Teaching of Elementary Mathematics.	W. J. G.	57
K.	Spooner.	The Elements of Geometrical Drawing.	W. J. Dobbs.	209
R.	Slate.	The Principles of Mechanics.	G. B. Mathews.	52
K.	Stevens.	v. Hall.		

SUBJECT.	AUTHOR.	BOOK.	REVIEWER.	PAGE.
R; S; T.	Stokes.	Mathematical and Physical Papers.	<i>G. B. Mathews.</i>	204
I.	Stolz.	v. Gmeiner.		52
K.	Taylor, C.	The Elementary Geometry of Conics.	<i>F. S. Macaulay.</i>	313
I.	Taylor, C. M.	Exercises in Arithmetic.	<i>W. J. G.</i>	394
X.	Taylor, G.	Logarithms.	<i>E. M. Langley.</i>	208
A.	Thomson, W.	Algebra.	<i>W. J. G.</i>	34
K.	„	W. v. Blaikie.		
I.	Timpany.	Arithmetic of Physics and Chemistry.	<i>W. J. G.</i>	395
A.	Tuckey.	Examples in Algebra.	<i>W. J. G.</i>	230
A.	Varley.	Algebra for Junior Students.	<i>W. J. G.</i>	146
K.	„	Euclid I. and II.	<i>W. J. G.</i>	116
C.	Vogt.	Éléments de Mathématiques Supérieures.	<i>W. J. G.</i>	123
X.	Vose.	A Graphic Method for certain questions in Arithmetic and Algebra.	<i>W. J. G.</i>	293
K.	Warren.	Experimental and Theoretical Geometry.	<i>E. M. Langley.</i>	315
T.	Weber, H.	Die Partiellen Differential-Gleichungen der mathematischen Physik. Vol. I.	<i>G. B. Mathews.</i>	51
T.	„	„ Vol. II.	<i>G. B. Mathews.</i>	113
T.	Weber, R.	Problems in Electricity.	<i>W. J. G.</i>	217
K.	Wells.	The Essentials of Plane and Solid Geometry.	<i>E. M. Langley.</i>	293
K.	Wentworth.	Plane and Spherical Trigonometry.	<i>W. J. G.</i>	318
C.	Whittaker.	A Course of Modern Analysis.	<i>G. B. Mathews.</i>	290
J.	Whitworth.	Choice and Chance.	<i>W. J. G.</i>	146
V.	Wölffing.	Mathematischer Bücherschatz.	<i>W. J. G.</i>	399
I.	Workman.	The Tutorial Arithmetic.	<i>E. M. Langley.</i>	206
I.	„	The School Arithmetic.	<i>W. J. G.</i>	391
R.	Worthington.	Dynamics of Rotation.	<i>G. H. Bryan.</i>	264
K.	Wyatt.	Preliminary Geometry.	<i>E. M. Langley.</i>	369
C.	Young.	v. Linebarger.		
V.	„	The Teaching of Mathematics in the Higher Schools of Prussia.	<i>E. M. Langley.</i>	114
V.	Zeuthen.	Histoire des Mathématiques dans l'Antiquité et le Moyen Age.	<i>W. J. G.</i>	248
R.	Ziwet.	An Elementary Treatise on Theoretical Mechanics.	<i>W. J. G.</i>	230

MISCELLANEA.

SUBJECT.	BOOK.	REVIEWER.	PAGE.
V.	Annuaire pour l'an 1901.	W. J. G.	57
	„ „ 1903.	W. J. G.	294
V.	Proceedings of the Edinburgh Mathematical Society.	G. B. Mathews.	114
V.	Woolwich Exam. Papers 1891-1900.	W. J. G.	146
V.	Comptes Rendus du deuxième congrès international des mathématiciens.	W. J. G.	249
V.	The Schoolmasters' Year Book and Directory.	W. J. G.	294

PROBLEMS AND SOLUTIONS.

No.		PAGE.	No.		PAGE.
28	K. 1. c.	22	175	P. 2. a.	13
32	P. 3. b.	23	252	I. 19. a.	13
37	R. 4. a.	23	271	J. 1. a.	14
39	A. 1.	23	287	D. 1. a.	15
42	K. 5. a.	23	306	J. 1. a.	15
51	K. 15. a.; I. 1.	37	307	I. 2. b.	15
53	K. 9. b.	37	311	I. 1.	16
54	K. 11. b.	37	320	(a) L ¹ . 16. b.	16
56	K. 10. e.	38		(b) L ¹ . 4. c.	16
58	K. 2. a.	38	327	B. 2. a.	17
60	K. 12. b.	76	338	D. 2. b. β .	296
61	K. 8. d.	39	344	L ¹ . 10. b.	17
62	A. 1. a.	39	345	L ¹ . 2. b.	18
63	K. 8. e.	39	347	P. 3. b.	18
64	K. 10. e.	39	350	K. 2. a.	19
65	K. 10. e.	39	356	I. 17. a.	124
66	K. 20. d.	39	360	K. 10. e.	19
68	R. 1. d.	40	362	K. 10. e.	20
72	K. 10. e.	40	363	L ¹ . 3. e.	20
75	L ¹ . 11. b.	40	364	K. 5. a.	102
78	L ¹ . 17. a.	40	365	K. 20. d.	20
81	A. 1. c. β .	77	366	K. 3. b.	21, 42
82	A. 1. a.	77	367	J. 2. e.	21
83	A. 1. a.	77	369	K. 5. d.	125
87	L ¹ . 10. a.	77	372	K. 13. a.	21
92	K. 8. b.	77	375	K. 20. e.	102
93	K. 8. b.	78	380	L ¹ . 11. c.	43
95	A. 1. a.	78	382	D. 2. d.	297
96	A. 2. b.	78	383	D. 2. d.	43
97	A. 1. a.; I. 1.	78	384	L ¹ . 17. e.	62
98	J. 2. f.	40, 78	385	K. 13. a.	22
99	K. 20. c. a.	79	386	L ¹ . 3. c.	22, 43
100	K. 20. d.	79	387	K. 17. e.; 20. f.	127
101	K. 20. e. a.	79	388	K. 11. e.	128
102	K. 20. e. a.	101	390	K. 13. a.	163
106	L ¹ . 2. c.	101	391	K. 20. e.	180
108	L ¹ . 10. b.	101	392	L ¹ . 4. c.	192
109	L ¹ . 7. d.	101	393	M ¹ . 2. a.	232
110	L ¹ . 5. a.	102	394	P. 2. a.	193
111	R. 4. c.	41	396	I. 25. b.	193
126	K. 8. e.	41	399	L ¹ . 17. e.; 19. d.	194
142	M ⁴ . a.	42	400	K. 2. d.; 5. d.	251

No.		PAGE.	No.		PAGE.
402	B. 1. a.	233	419	R. 9. b.	273
403	K. 4.	234	421	D. 2. d.	321
405	K. 12. b. a.	234	424	R. 1. a.	299
406	K. 3. a.	270	425	K. 2. c.	274
407	L ¹ . 4. a. a.	235	426	L ¹ . 16. a.	364
408	L ¹ . 14. a.	235	427	K. 12. b.	365
409	M ¹ . 3. j.	195	428	R. 9. b.	366
410	L ¹ . 4. c.; M ¹ . 3. b.	297	432	A. 3. a.	366
411	A. 3. K.	235	433	K. 13. c.	385
412	L ¹ . 2.; 14. a.	298	434	K. 21. β .	385
413	K.	252	435	L ¹ . 10. a.	386
415	K. 9. b.	272	450	K. 20. e.	274
416	M ¹ . 2. a.	273	451	J. 1. a. b.	274
417	L ¹ . 5.; 17. a.	298	453	L ¹ . 17. c.	275
418	R. 7. b.	321	463	L ¹ . 17. e.	386

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 1.

JANUARY, 1901.

One Shilling Net.

THE ANNUAL GENERAL MEETING of the Association will be held on Saturday, January 19, 1901, at 2 p.m., at King's College, Strand, W.C. The President, Sir ROBERT BALL, F.R.S., will read a paper on "Some Contributions to Geometry from recent Dynamical Work." Professor M. J. M. HILL, F.R.S., will introduce a discussion on the teaching of Proportion in Geometry.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

Agent in France :—A. HERMANN, Librairie Scientifique,
Rue de la Sorbonne 8, Paris.

Problems and Solutions should be sent to

W. E. HARTLEY, 498 Union Street, Aberdeen.

All other correspondence should be addressed to

W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

CONTENTS.

	PAGE
TWO ILLUSTRATIONS OF ELIMINATION; R. F. Davis, M.A., ...	1
NOTE ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS; J. Brill, M.A., ...	3
REVIEWS:	
Roberts' <i>Dynamics</i> , etc.; W. H. Macaulay, M.A., ...	5
ON MARKING EUCLID PAPERS; C. S. J., ...	8
MATHEMATICAL NOTES; T. J. I'a Bromwich, M.A.; E. Budden, M.A., B.Sc.,	9
PROBLEMS, ...	12
SOLUTIONS, ...	13
BOOKS, ETC., RECEIVED, ...	24

Will appear shortly:

Areal Coordinates and their Applications. (G. D. Muggeridge, B.A.)

The Slide Rule. (Prof. F. R. Barrell, M.A., B.Sc.)

Reviews:

The Calculus for Chemists, Engineers, etc. (Prof. G. B. Mathews, F.R.S.)

Charnock's *Graphic Statics*, etc.; W. J. Dobbs, M.A.

Forsyth's *Differential Equations*, Part II.; Prof. W. H. Lloyd-Tanner, F.R.S., D.Sc.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. *No. 8 is out of print.* A few numbers may be obtained from *the Editor* at the following prices, post free:

No. 7, 2s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 10s. 6d.

These will soon be out of print and very scarce.

THE *Annals of Math* MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON :
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

TWO ILLUSTRATIONS OF ELIMINATION.

1. If A, B, C, G are four given points in a plane, it is possible to place masses x, y, z at A, B, C respectively, so that the centroid of the system is at G . This, it is to be observed, can only be done in one way as x, y, z must be proportional to the areal (or "barycentric") co-ordinates of G with respect to the triangle ABC .

If, then, P is any variable point in the same plane,

$$\Sigma xPA^2 - PG^2 \Sigma x$$

is independent of P (the "moments of inertia" theorem due to Euler).

2. Applying this theorem to each of the points A, B, C , and G (whose distances from A, B, C are p, q, r respectively)

$$\begin{aligned} +c^2y + b^2z - (x+y+z)p^2 &= K, \\ c^2x + a^2z - (x+y+z)q^2 &= K, \\ b^2x + a^2y - (x+y+z)r^2 &= K, \\ p^2x + q^2y + r^2z &= K, \end{aligned}$$

also
$$x + y + z - (x+y+z) = 0.$$

3. Eliminating $x, y, z, -(x+y+z)$, and $-K$ from these equations:

$$\begin{vmatrix} 0, & c^2, & b^2, & p^2, & 1 \\ c^2, & 0, & a^2, & q^2, & 1 \\ b^2, & a^2, & 0, & r^2, & 1 \\ p^2, & q^2, & r^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

or $a^2p^4 + b^2q^4 + c^2r^4 = (b^2 + c^2 - a^2)(q^2r^2 + a^2p^2) + \dots + \dots - a^2b^2c^2,$

the relation connecting p, q, r which can be otherwise obtained as in Todhunter's *Spherical Trigonometry*, and Levett and Davison's *Trig.*, p. 172.

4. In the *Educational Times*, Reprint Vol. LXVII., Q. 13424, p. 113, I have shewn that the moments of inertia theorem takes, in the case of three points A, B, C only, a simpler form

$$\Sigma x \cdot PA^2 = (PG^2 - GT^2) \Sigma x,$$

where GT is the tangent from G to the circum-circle.

If, therefore, A, B, C, G be concyclic, $K=0$; and the relation becomes

$$\begin{vmatrix} 0, & c^2, & b^2, & p^2 \\ c^2, & 0, & a^2, & q^2 \\ b^2, & a^2, & 0, & r^2 \\ 1, & 1, & 1, & 0 \end{vmatrix} = 0,$$

or

$$ap \pm bq \pm cr = 0.$$

5. Now let A, B, C, D be four points in a plane at which masses x, y, z, u are respectively placed; G the centroid of the system.

From A, B, C, D as centres describe circles of radii a, b, c, d and from G as centre describe a circle of radius

$$\Sigma xa / \Sigma x (=g).$$

Then if P be the centre and r the radius of any variable circle,

$$\Sigma x(PA)^2 - (PG)^2 \Sigma x$$

is independent of both P and r ; (PA) being the length of the common tangent to the circles (P, r) and (A, a).

For the above expression

$$\begin{aligned} &= \Sigma x \{PA^2 - (r-a)^2\} - \{PG^2 - (r-g)^2\} \cdot \Sigma x \\ &= \Sigma x PA^2 - PG^2 \cdot \Sigma x + \text{terms in } r^2 \text{ and } r \text{ (which cancel out)} \\ &\quad + \text{constant} \\ &= \text{constant.} \end{aligned}$$

This relation is of the same form as that in (1).

6. Applying in a similar manner the theorem to A, B, C, D , and G , we obtain the relation between the common tangents to five circles as given in Salmon:

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1, & 1 \\ 1, & 0, & (12)^2, & (13)^2, & (14)^2, & (15)^2 \\ 1, & (12)^2, & 0, & (23)^2, & (34)^2, & (25)^2 \\ 1, & (13)^2, & (23)^2, & 0, & (34)^2, & (35)^2 \\ 1, & (14)^2, & (24)^2, & (34)^2, & 0, & (45)^2 \\ 1, & (15)^2, & (25)^2, & (35)^2, & (45)^2, & 0 \end{vmatrix} = 0.$$

7. It is to be observed that A, B, C, D, G being centres of given circles of radii a, b, c, d, g respectively, we can arrange the masses x, y, z, u to be placed at A, B, C, D so that G shall be the centroid of the system in an infinite number of ways.

We can, therefore, so arrange them that not only is G the centroid but also $\Sigma xa = g\Sigma x$ as well. The foregoing proof is therefore perfectly general.

R. F. DAVIS.

NOTE ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

1. Let $x^3 + px^2 + qx + r = 0$

be a cubic equation, whose roots are α, β, γ . Then, defining the quantity λ by means of the equation $\lambda^3 = 1$, we have

$$\begin{aligned} (\alpha + \lambda\beta + \lambda^2\gamma)^2 &= \alpha^2 + \beta^2 + \gamma^2 + 6\alpha\beta\gamma + 3\lambda(\beta^2\gamma + \gamma^2\alpha + \alpha^2\beta) \\ &\quad + 3\lambda^2(\beta\gamma^2 + \gamma\alpha^2 + \alpha\beta^2) \\ &= A + \lambda B + \lambda^2 C. \end{aligned}$$

Since this is true for all values of λ that satisfy the equation $\lambda^3 = 1$; we have, ω being one of the imaginary cube roots of unity, the three equations

$$\begin{aligned} \alpha + \beta + \gamma &= (A + B + C)^{\frac{1}{3}}, \\ \alpha + \omega\beta + \omega^2\gamma &= (A + \omega B + \omega^2 C)^{\frac{1}{3}}, \\ \alpha + \omega^2\beta + \omega\gamma &= (A + \omega^2 B + \omega C)^{\frac{1}{3}}. \end{aligned}$$

From these equations it follows that

$$\begin{aligned} 3\alpha &= (A + B + C)^{\frac{1}{3}} + (A + \omega B + \omega^2 C)^{\frac{1}{3}} + (A + \omega^2 B + \omega C)^{\frac{1}{3}}, \\ 3\beta &= (A + B + C)^{\frac{1}{3}} + \omega^2(A + \omega B + \omega^2 C)^{\frac{1}{3}} + \omega(A + \omega^2 B + \omega C)^{\frac{1}{3}}, \\ 3\gamma &= (A + B + C)^{\frac{1}{3}} + \omega(A + \omega B + \omega^2 C)^{\frac{1}{3}} + \omega^2(A + \omega^2 B + \omega C)^{\frac{1}{3}}. \end{aligned}$$

Thus, if A, B, C can be expressed in terms of the coefficients of the given cubic equation, we have a solution of the said equation. Now A is a symmetrical function of the roots of the said equation, and may therefore be so expressed. In fact we have

$$A = -p^2 + 3pq - qr.$$

Writing $U = \beta^2\gamma + \gamma^2\alpha + \alpha^2\beta$,
and $V = \beta\gamma^2 + \gamma\alpha^2 + \alpha\beta^2$,

we see that $U + V$ is a symmetrical function of the roots. In fact we have

$$U + V = -pq + 3r. \dots\dots\dots(1)$$

Also
$$\begin{aligned} U - V &= \beta\gamma(\beta - \gamma) + \gamma\alpha(\gamma - \alpha) + \alpha\beta(\alpha - \beta) \\ &= -(\beta - \gamma)(\gamma - \alpha)(\alpha - \beta). \end{aligned}$$

This shows that $(U - V)^2$ is a symmetrical function of the roots; and, as $U + V$ is of the same character, it follows that UV must be so also. In fact we have

$$\begin{aligned} UV &= \beta^3\gamma^3 + \gamma^3\alpha^3 + \alpha^3\beta^3 + 3\alpha^2\beta^2\gamma^2 + \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2) \\ &= p^2r + q^3 - 6pqr + qr^2. \dots\dots\dots(2) \end{aligned}$$

From equations (1) and (2) the values of U and V can be readily obtained by a known method. Thus the values of B and C , in terms of the coefficients of the equation, can be found, and the solution completed.

2. Next suppose that we have a biquadratic equation

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

whose roots are $\alpha, \beta, \gamma, \delta$. This time we will employ two quantities, λ and μ , defined by the equations

$$\lambda^3 = 1, \quad \mu^2 = 1.$$

$$\begin{aligned} \text{We have } (a + \lambda\beta + \mu\gamma + \lambda\mu\delta)^2 &= a^2 + \beta^2 + \gamma^2 + \delta^2 + 2\lambda(a\beta + \gamma\delta) \\ &\quad + 2\mu(a\gamma + \beta\delta) + 2\lambda\mu(a\delta + \beta\gamma) \\ &= A + \lambda B + \mu C + \lambda\mu D. \end{aligned}$$

Now this equation is satisfied by any pair of possible values of λ and μ . We have, therefore, four cases, viz.,

$$\begin{aligned} \lambda &= 1, & \mu &= 1, \\ \lambda &= -1, & \mu &= 1, \\ \lambda &= 1, & \mu &= -1, \\ \lambda &= -1, & \mu &= -1. \end{aligned}$$

$$\begin{aligned} \text{Thus we obtain } a + \beta + \gamma + \delta &= (A + B + C + D)^{\frac{1}{2}}, \\ a - \beta + \gamma - \delta &= (A - B + C - D)^{\frac{1}{2}}, \\ a + \beta - \gamma - \delta &= (A + B - C - D)^{\frac{1}{2}}, \\ a - \beta - \gamma + \delta &= (A - B - C + D)^{\frac{1}{2}}. \end{aligned}$$

Thus, if A, B, C, D can be expressed in terms of the coefficients of the equation, we readily obtain the solution in the following form :

$$\begin{aligned} 4a &= (A + B + C + D)^{\frac{1}{2}} + (A - B + C - D)^{\frac{1}{2}} + (A + B - C - D)^{\frac{1}{2}} \\ &\quad + (A - B - C + D)^{\frac{1}{2}}, \\ 4\beta &= (A + B + C + D)^{\frac{1}{2}} - (A - B + C - D)^{\frac{1}{2}} + (A + B - C - D)^{\frac{1}{2}} \\ &\quad - (A - B - C + D)^{\frac{1}{2}}, \\ 4\gamma &= (A + B + C + D)^{\frac{1}{2}} + (A - B + C - D)^{\frac{1}{2}} - (A + B - C - D)^{\frac{1}{2}} \\ &\quad - (A - B - C + D)^{\frac{1}{2}}, \\ 4\delta &= (A + B + C + D)^{\frac{1}{2}} - (A - B + C - D)^{\frac{1}{2}} - (A + B - C - D)^{\frac{1}{2}} \\ &\quad + (A - B - C + D)^{\frac{1}{2}}. \end{aligned}$$

Now A is a symmetrical function of the roots, and we have

$$A = p^2 - 2q.$$

The solution of the equation, therefore, depends upon the expression of the three quantities

$$a\beta + \gamma\delta, \quad a\gamma + \beta\delta, \quad a\delta + \beta\gamma$$

in terms of the coefficients of the equation. Now these three quantities clearly form a group, i.e. they are such that any interchanges effected among the quantities $\alpha, \beta, \gamma, \delta$ leave us with the same three quantities. Thus it is clear that if we form a cubic equation with these three quantities for roots, its coefficients will be expressible in terms of the coefficients of the given equation. In fact, writing U, V, W for the three quantities, we have

$$\begin{aligned} U + V + W &= \Sigma a\beta = q, \\ VW + WU + UV &= \Sigma a^2\beta\gamma = pr - 4s, \\ UVW &= \Sigma a^2\beta\gamma\delta + \Sigma a^2\beta^2\gamma^2 \\ &= p^2s - 4qs + r^2. \end{aligned}$$

Thus the auxiliary cubic, whose roots are U, V, W , is

$$x^3 - qx^2 + (pr - 4s)x - p^2s + 4qs - r^2 = 0.$$

The solution of this equation will give U, V, W , and therefore B, C, D in terms of the coefficients of the biquadratic.

J. BRILL.

REVIEWS.

A Treatise on Elementary Dynamics, dealing with relative motion mainly in two dimensions, by H. A. ROBERTS, M.A. Macmillan & Co., 1900, pp. xi. + 258.

The Student's Dynamics, comprising Statics and Kinetics, by G. M. MINCHIN, M.A., F.R.S. George Bell & Sons, 1900, pp. xii. + 258.

Elementary Mechanics of Solids, by W. T. A. EMTAGE, M.A. Macmillan & Co., 1900, pp. viii. + 333.

Mr. Roberts's book is the most ambitious of these three. It does not profess to belong to the most elementary class of text-books on the subject; and the author, while restricting himself to the scope indicated by the title, makes a serious attempt to make good the ground he occupies, and to tackle the subject with a sense of responsibility towards students of science. A notable feature of the book is the way in which, in the discussion of dynamical principles, and the application of them to practical questions, some care is taken to specify the base relative to which motion is reckoned. Thus an old-established source of perplexity to students of the subject is to a considerable extent avoided. The author adopts the term "Galileo's axes" as the name for a base attached to the surface of the earth; and, by the use of this term, is able to call attention, in a convenient way, to the fact that the ordinary theory applied, without correction, to motion relative to the earth is only approximate, without introducing any complicated calculations on the subject at too early a stage. An extension to the treatment of motion relative to other bases, such as a base suitable for the whole solar system, which Mr. Roberts calls "Newton's axes," is easily and naturally made. This way of developing the subject seems to be satisfactory, and Mr. Roberts's treatment is on the whole decidedly good, though no doubt some trifling improvements could be suggested. The establishment of mass to express acceleration-ratios is well done.

Another important feature is a certain amount of use of vector notation, which is quite suitable to a book of this class, and helps it a great deal. It is a curious thing how slow writers of elementary text-books have been to take advantage of the attractive features presented by such treatment, which, kept within due limits, is certain to interest an intelligent boy. We notice that Mr. Roberts is apt to refer to the motion of a point relative to another *point*, where motion relative to a certain base or axes is what is really meant. This form of words is suitable, for the sake of brevity, on some occasions; but we think it occurs too often, and is the cause of some obscurity. It is to be found also in Professor Minchin's book. Mr. Roberts makes some use of the hodograph for transforming a theorem about radius and velocity into one about velocity and acceleration, and calls attention to the scope of this convenient method. The book contains a large number of useful examples.

Professor Minchin's book is intentionally more elementary. Some parts of it are not without merit. There is a good deal of clear

explanation, helped by arithmetical examples. But in some respects it is not a satisfactory book. Some necessary explanations are altogether omitted. We find no explanation of acceleration except for the case of rectilinear motion; and yet the direction of acceleration, and finally normal acceleration are dealt with. The treatment of the fundamental principles of dynamics is incomplete and not satisfactory. And throughout the book occasional inaccurate statements are to be found; such as, on p. 94, that a force can do negative work only if some other force is acting on the body, and, on p. 244, that Hooke's law holds for a steel bar with extension $1/100$.

Anyone would suppose from the preface that Professor Minchin had adopted the course of defining force in terms of acceleration. But, after reading the book, it is not easy to make out what the procedure is which the preface refers to. As a matter of fact, Professor Minchin, though he refers to acceleration for the specification of a unit, a dyne—employs the method of defining the measurement of force by the superposition of conditions, and treats its relation to acceleration as an experimental result. He refers to spiral springs in this connection to give definiteness to the explanations and diagrams. We think that this course has advantages within a certain limited range of treatment, and we do not altogether condemn it; but the way in which Professor Minchin connects forces, indicated by springs, with acceleration and masses is incomplete and not clear. There is no specification of the base relative to which the accelerations which he refers to are to be measured, nor is the question raised as to how far the specification of a base is needed. Weight is adopted as the test of equality of masses of different substances, a retrograde procedure, and one applicable only to terrestrial bodies. On p. 16 we find it tacitly assumed that equal masses, tested in this way, of all substances will give the same dyne. Then we are told to assume that, for a given mass, acceleration is proportional to force measured by superposition of springs. In fact there are not sufficient indications of the character of the steps which lead to the result which is aimed at, $P = ma$.

Professor Minchin says, in his preface, that he has "made another effort to banish that extremely misleading term *centrifugal force*"; so we naturally look to see what he has done to accomplish this. On p. 211 he says, "as regards the tendency of a revolving body to fly outwards from the centre, no such thing exists." He can hardly expect any reader who has ever seen the governor of a steam-engine, or the behaviour of a system attached to a whirling table, to agree with this statement. Surely this is not the way to approach the subject. In the whole of his discussion Professor Minchin does not even mention the two points of real importance, namely: (1) that centrifugal force is exhibited only in cases of rotation relative to the directions of the fixed stars, and (2) that it is condemned as a force by the law of action and reaction. Indeed, we actually find, on the same page, 211, the statement that, in the case of a heavy body on the earth, the earth experiences a pull which is "the exact equal and opposite" of the weight of the body. Now, the earth is an example of a whirling table, and the component of weight, which is due to the rotation, is a good example of the cases in which centri-

fugal force (by whatever name it is called) is useful. Professor Minchin's treatment, in fact, seems to us to be calculated to land his readers in hopeless perplexity.

On p. 18 Professor Minchin has an article on the nature of weight. He discusses the law of gravity as applied to a sphere, and how gravitation varies at different distances from the centre of the earth, and tells us that, in connection with a body on the surface of the earth, it is called the weight of the body. He goes on to say that weight, thus defined, varies at different places on the earth because the earth is not quite spherical. He then tells us how this variation of weight might be measured, and that the greatest amount of variation is $\frac{1}{2}$ per cent. A reader of this article might be surprised to learn that, as a matter of fact, though the total variation of weight between the equator and the pole is about $\frac{1}{2}$ per cent., only about one-third of this variation is due to the law of gravitation and the flattening of the poles, the remaining two-thirds being due to centrifugal force. Mr. Roberts deals to some extent with the same topic, but the point which he discusses in detail is the centrifugal force part of the variation. He might very well have mentioned the part due to the law of gravitation, and it was rather a pity to omit a reference to this. But his statements on the subject are clear enough; a calculation is made on the assumption that the earth is spherical, and he mentions both the whole variation of weight and the part due to centrifugal force. We do not, however, think he is right in suggesting, as he does in one of the examples on p. 122, that a railway train travelling west is heavier than when travelling east; this appears to be introducing into weight an element which does not belong to it. In Mr. Emtage's book we find both parts of the variation of gravity explained in general terms.

A peculiarity of Professor Minchin's book is that he adopts (see p. 56) the plan of defining his symbols for lengths, forces, etc., as representing concrete lengths, forces, etc., and not merely the numerical values of them. Accordingly he explains the necessary homogeneity of equations in symbols of one kind, not as arising out of certain liberty which we have reserved for ourselves as to choice of units, which is the usual point of view, but as arising out of the impossibility of two concrete things of different kinds being equal. He says that the cube of a length is a volume, but does not attempt to construct a complete calculus for symbols treated in this way. This is starting on an impracticable enterprise, and it is hardly necessary to say that it is not carried through consistently. Every case in which a symbol for force or velocity is equated to a mere number is a violation of the system, and there are many such cases throughout the book. The system appears to us to be a very bad one, introducing, as it does, quite unnecessarily, difficulties of interpretation of combinations of symbols.

Mr. Emtage's book, we are told in the preface, follows a South Kensington syllabus, but contains some additional matter. A notable feature of it is the experimental illustrations, which are carefully described, with a view to their being performed with very simple apparatus. Another peculiarity of it is the way in which the subject

is developed by the aid of independent experimental, so-called, proofs of the parallelogram of forces, equation of moments, and so on, making a number of different points of departure. In some cases we are given theoretical proofs afterwards, but we cannot find even a mention of the possibility of any theoretical investigation of the parallelogram of forces. We gather that this peculiarity is derived from the South Kensington syllabus. The book seems to be a good one for the purpose for which it is intended, and likely to be useful.

W. H. MACAULAY.

[v.] ON MARKING EUCLID PAPERS.

THE following is a summary of the answers received on this question—some from correspondents of considerable experience as examiners.

Question.	Largest Deduction.	Least Deduction.	Average of all Replies.
1. a.	20	5	10
1. b.	90	5	17·2
2. a.	75	10	24·5
2. b.	10	0	1 nearly.

The very considerable differences of opinion indicated above, probably arise from the fact that some examiners, starting from a perfect answer, deduct marks for every defect in order to do justice to the perfect answer, while others, starting from the average pupil, desire to give a few marks for anything approaching to a correct answer.

Again, it is sometimes doubtful whether a particular answer discloses a mere slip or a grave mistake in principle. Some examiners give the benefit of the doubt to the answer, others to those who have given the correct answer. *On the whole* the replies seem to indicate that the most experienced examiners take the most lenient view of mistakes.

Some correspondents who desire the overthrow of Euclid see an argument in favour of their view. The fact is, that in marking or teaching Euclid we are really dealing with *Geometry and Logic*. Hence some who denounce Euclid, say they could teach or examine Geometry better without him, which, no doubt, in a sense is true. Also others who, seeing no hope for the inclusion of Logic apart from Euclid in our scheme of education, and regarding Logic, the art of reasoning correctly on unfamiliar matters, as a very important part of education, cling to Euclid, with all the concomitant disadvantages.

The question appears to show very clearly the necessity in important competitive examinations of having the scale of marking settled by two or more examiners in conjunction, and the great assistance which would be furnished by a *visd voce* in deciding whether a given reply involves a slip or a mistake. Competitive mathematical examinations largely govern the mathematical education of the country, and in some respects (notably in fostering the craze for short methods and dodges) have a very bad effect. Every examiner and teacher knows how very large the probable error of the marks is in any examination conducted by *individuals* (as are all Civil Service Commission examinations), but this is not so well known as it ought to be to the general public. The assumed infallibility of the examination test has led to the gradual rejection of all other modes of selection, and to a general acquiescence in the effects produced by examinations.

C. S. J.

MATHEMATICAL NOTES.

92. [K. 13. a.] *Solution of Problem 372, and notes suggested thereby.*

No solution of this having yet appeared, I add the following analytical notes to my previous statement.

The general (finite) displacement of a rigid body by a screw about the axis of z is given by

$$x' = ax, \quad y' = \frac{1}{a}y, \quad z' = z + \beta,$$

where

$$x = X + iY, \quad y = X - iY;$$

X, Y, z are a rectangular Cartesian set of coordinates and a, β are constants, of which β is real and the absolute value of a is unity; in fact $a = e^{i\phi}$, if ϕ be the angle of rotation. Then a point midway between $(x'y'z')$ and (xyz) is

$$x_0 = \frac{1}{2}(1+a)x, \quad y_0 = \frac{1}{2a}(1+a)y, \quad z_0 = z + \frac{1}{2}\beta,$$

so that if (xyz) is in a given plane

$$Ax + By + Cz + D = 0,$$

then $(x_0y_0z_0)$ is in another plane.

The plane bisecting at right angles the line joining (xyz) , $(x'y'z')$ is

$$(x' - x)(\eta - y_0) + (y' - y)(\xi - x_0) + 2(z' - z)(\zeta - z_0) = 0,$$

(ξ, η, ζ) being current coordinates in the plane. This reduces to

$$(1-a)\left(\frac{1}{a}\xi y - x\eta\right) + \beta(2\zeta - 2z - \beta) = 0.$$

Hence with (xyz) in the given plane, the latter plane contains the fixed point (O)

$$\frac{-(1-a)\eta}{A} = \frac{(1-a)\xi}{aB} = \frac{-2\beta}{C} = \frac{\beta(2\zeta - \beta)}{D},$$

which completes the solution of the problem.

If O has been displaced from the point O' , we have

$$\xi' = \frac{1}{a}\xi = -\frac{2\beta}{1-a} \frac{B}{C}, \quad \eta' = a\eta = \frac{2\beta a}{1-a} \frac{A}{C}, \quad \zeta' = \zeta - \beta = -\frac{\beta}{2} - \frac{D}{C},$$

and with these values it can be proved (just as above) that the plane

$$Ax + By + Cz + D = 0$$

bisects OO' at right angles; this is the property of O stated in my last note (*Gazette*, Oct., 1900).

If we desire to evaluate the *velocities* due to an instantaneous screw about the axis of z , we have to make β, ϕ infinitesimal, and then we find

$$\frac{dx}{dt} = i\theta x, \quad \frac{dy}{dt} = -i\theta y, \quad \frac{dz}{dt} = \gamma,$$

where $\theta = \frac{d\phi}{dt}$, $\gamma = \frac{d\beta}{dt}$, so that both are real; and the pitch of the screw is γ/θ .

In these coordinates we can prove very easily a theorem due to Cayley, that every point $(x_0y_0z_0)$ can move instantaneously along the line joining $(x'y'z')$, (xyz) for all positions of (xyz) . For, in terms of $(x_0y_0z_0)$,

$$x' - x = 2\frac{a-1}{a+1}x_0, \quad y' - y = -2\frac{a-1}{a+1}y_0, \quad z' - z = \beta,$$

and thus the point (x_0, y_0, z_0) can be screwed along this line by an infinitesimal screw about the axis of z of pitch $\frac{\beta}{2} \frac{\alpha+1}{\alpha-1} = \frac{\beta}{2} \cot \frac{\phi}{2}$ (using the expressions for the velocities just found). So that, in particular, if the original motion be infinitesimal, this value of the pitch is (β/ϕ) as it ought to be.

In concluding, it may be well to point out wherein the three-dimensional results differ from the two-dimensional. In the two-dimensional case, we have only to ignore z and β in the above; so that the line bisecting at right angles the join of (xy) , $(x'y')$, or

$$(x' - x)(\eta - y_0) + (y' - y)(\xi - x_0) = 0,$$

reduces to

$$\frac{1}{\alpha} \xi y - x \eta = 0,$$

which passes through the origin whatever the point (x, y) may be; thus the origin is a marked point of the body which does not move in the displacement. This is of course a familiar property, and is only mentioned now for comparison with the other results.

If we assume that the usual conditions for orthogonality of lines, etc., can be extended to space of n dimensions, merely by adding on terms of the same type; I have proved that when n is *odd* there is no real point at a finite distance which does not move in a general displacement of a rigid body; but if n is *even*, there is always at least one such point (cf. R. F. Muirhead, *Proc. Edin. Math. Soc.*, Vol. xvi., p. 70). Of course by specializing the motion we can always keep a certain number of points fixed; but if n be odd, and a point is kept fixed, then a line is thereby also fixed.

Before leaving this subject, I should like to remark that the use of coordinates such as (x, y) above (*circular coordinates*) is often advantageous in geometrical problems that involve metrical relations; thus, properties of the epicycloids, etc., have been very simply obtained from these coordinates by Prof. F. Morley (*Am. Journal of Maths.*, Vols. XIII., XVI.): see also Vol. I., No. 2, of the *Transactions of the American Mathematical Society*.

T. J. P. A. BROMWICH.

93. [K.] Prof. Hill's Eucl. v. and vi., Definition of Ratios, and Incommensurables.

I am glad to have the opportunity of showing that my proposal of *measure* as the definition of ratio is free from the difficulties suggested by Prof. Hill* (Review of Prof. Hill's Eucl. v. and vi. Vol. I. No. 24, p. 411).

ART. 1. If A and B are commensurable so that $A = sC$, $B = rC$, then the measure, $A|B$ say, is the rational fraction s/r , since $A = s/r \cdot B$.

When A and B are incommensurable we find, as in the decimal system of weighing, a terminating decimal $\mu_n = \omega \cdot a_1 a_2 \dots a_n$, such that, the decimal μ'_n being the same as μ_n with $a_n + 1$ for a_n , and μ the true measure $A|B$,

$$A = \mu B > \mu'_n B < \mu'_n B.$$

The process is supposed to be carried far enough to show any required difference between μ and any different decimal ν .

ART. 2. *Theorem I.* If the measures $A|B$, $C|D$ are μ , ν , and if, r and s being integers, whenever $rA > sB$, $rC > sD$, and when $<$, then $A|B = C|D$.

* The less important points raised by Prof. Hill are passed over for lack of space, as well as a detailed account of measurement in Art. 1 justifying (a) in Art. 2.

Suppose $\mu > \nu$. Find (Prof. Hill's Prop vii.) integers r, s such that

$$\begin{aligned} r\mu &> s.1 > r\nu & \dots\dots\dots (a) \\ \therefore r\mu B &> sB, \text{ but } r\nu D < sD, & \dots\dots\dots (b) \\ \therefore rA &> sB, \text{ but } rC < sD, \text{ contrary to hypothesis;} \\ \therefore \mu &\nless \nu, \text{ similarly } \mu \nless \nu; \therefore \mu = \nu. \end{aligned}$$

Note i. Professor Hill uses the same argument (a) in his *Art. Phil. Trans.* Prop. 17; and $r\mu$ is intelligible as an operator in (b) just so far as it is intelligible as a number $> s$ in (a). Prop. vii is a direct result of Archimedes' Principle.

Note ii. Professor Hill's 3 sets of conditions $>$, $=$, and $<$, of Prop. viii. are reducible by Prop. vii. to the 2, $>$ and $<$, of this article.

ART. 3. *Theorem II.* If $A|B = \mu$, the scale $[A, B] = [\mu, 1]$.

For, r and s being integers,

$$\begin{aligned} \text{whenever} \quad rA &> sB, \quad r\mu B > sB \\ &\therefore r\mu > s.1. \\ \text{Similarly, whenever} \quad rA &< sB, \quad r\mu < s.1; \\ &\therefore [A, B] = [\mu, 1] \end{aligned}$$

Now when A, B are commensurable $= aN, bN$ say, Prof. Hill proves, Prop. xi.,

$$[A, B] = [a, b] = [a/b, 1];$$

and says (Art. 44), " a/b is taken as the measure of $a : b$ " i.e. $A : B$.

Similarly, we say, " μ is taken as the measure of $\mu : 1$ " (or $A : B$).

ART. 4. *Theorem III.* Hence, the measure $A|B$ is also the measure of the ratio $A : B$.

Among the advantages of defining ratio by this measure $A|B$ are:

(i.) We can prove the test (Art. 2), and it is therefore scientifically wrong to assume it; it is much easier to apply, and greater and less ratios become intelligible.

ii. The definition is that assumed in Trigonometry, etc.

iii. A fourth Proportional can be at once constructed.

ART. 5. As to the whole question of incommensurables:—I must point out that (i.) approximations being barred, Euclid's Book V. applies only to those kinds of magnitude for which a 4th Proportional can be found without approximation; these are, rational numbers, and certain figures in Euclidean space; and (ii.) it is significant that in general* when the magnitudes A, B , only are given, the scale $[A, B]$ and Euclid's ratio $A : B$ are indeterminate; since, however, many terms of the multiple order $1.B, 1.A, 2B$, etc. have been found, the number of possible continuations, and therefore of possible scales or ratios, is still infinite.†

Arts. 1, 2 above, however, show that the difficulties are largely imaginary, and are in any case contained in Archimedes' Principle, assumed in Euclid's postulates, in V., Def. 5, 7, and in Prof. Hill's Prop. vii. Euclid's and Prof. Hill's systems are therefore equally guilty with others. The merit of Euclid's test lies in applying the method of limits once for all.

I ought to say that I am deeply indebted to the suggestiveness of Prof. Hill's Book, and in particular to Prop. vii., for much of the above note and indeed for the suggestion of the 'measure' definition.

E. BUDDEN.

* The exception rational numbers.

† For instance, to the 3rd term given above, A is only known to be some magnitude between $1.B$ and $2.B$; $\therefore$ the number of possible continuations is infinite, and so on.

PROBLEMS.

[Much time and trouble will be saved the Editor if (even tentative) solutions are sent with problems by their Proposers.]

400. [K. 2. d; 5. d.] A variable straight line meets two fixed straight lines Ox, Oy in A and B , so that the sum or difference of OA, OB is constant: find the envelope of the Euler circle of the triangle OAB , and the locus of its Lemoine point and the three associated points.

Prove also that the circumcircle of OAB passes through another fixed point besides O , and that its Euler line also passes through a fixed point.

E. N. BARISIEN.

401. [I. 3.] Prove that

$$3 \cdot 10^n - 7^{2n-n} \equiv 0 \pmod{23}$$

for all values of n . Find a process for obtaining similar modular equations [$a \cdot B^n - c \cdot D^{2n-n} \equiv 0 \pmod{P}$] for other primes. Write the equation in its simplest form and find values for the letters when $P=97$.

R. W. D. CHRISTIE.

402. [B. 1. a.] Prove that the determinant

$$\begin{vmatrix} a_0 & a_1x & a_2x^2 & \dots & a_nx^n \\ a_nx^n & a_0 & a_1x & \dots & a_{n-1}x^{n-1} \\ a_{n-1}x^{n-1} & a_nx^n & a_0 & \dots & a_{n-2}x^{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1x & a_2x^2 & a_3x^3 & \dots & a_0 \end{vmatrix}$$

is a polynomial of order n in x^{n+1} .

F. S. MACAULAY.

403. [K. 4.] Construct geometrically a right-angled triangle having given the base b and the rectangle contained by the hypotenuse and the perpendicular.

ISAAC H. TURRELL.

404. [J. 2. c.] Material sufficient to make a solid sphere of radius unity is divided at random into two parts, each of which is made into a solid sphere: show that the expectation of the sum of the radii is $\frac{3}{2}$.

If the material had been divided into three parts, the expectation would be $\frac{7}{4}$; if into four parts, it would be $\frac{5}{2}$, and so on.

W. A. WHITWORTH.

405. [K. 12. b. a.] Draw three circles mutually tangent, which shall also touch a given straight line at given points.

C. E. YOUNGMAN.

406. [K. 3. a.] OAB is an isosceles triangle, base AB , and PAB is any other triangle on the same base and in the same plane: prove that

$$4 \cdot OA^2 \cdot PA \cdot PB \cdot \cos \frac{1}{2}(AOB - APB) + AB^2 \cdot OP^2 = OA^2(PA + PB)^2. \quad (C.)$$

407. [L. 4. a. a.] S, H are the foci of an ellipse, T an external point; SM and HN perpendicular respectively to ST and HT meet the adjacent tangents from T to the ellipse in M and N ; SN and HM intersect in L : shew that the perpendicular from T on SH passes through L . (C.)

408. [L. 14. a.] If the intersections of opposite sides of a circum quadrilateral of an ellipse lie on a coaxial and similar ellipse, shew that one pair of opposite vertices lie on another coaxial and similar ellipse. (C.)

409. [M. 3. j.] Lines are drawn from every point of a curve making an angle β with the tangent at the point: if $(\bar{x}, \bar{y})$ be the centre of curvature at a point (x, y) , and (ξ, η) the point corresponding to (x, y) on the envelope of the above lines, find ξ and η in terms of $x, y, \bar{x}, \bar{y}$, and β . (C.)

SOLUTIONS.

UNSOLVED QUESTIONS.—171, 275, 279, 283, 285, 326, 336-8, 341, 349, 356, 369, 370, 373, 376-82, 387, 389-99.

Solutions to these, or other questions to which no solution has yet been published, and to 400-9 should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn to a small scale on a separate sheet.

175. [P. 2. a.] *When a conic S is reciprocated with respect to any point O into a conic S' , prove that the reciprocals of the four foci of S are the two pairs of common chords of S' and a point-circle at O .* R. P. ROYSTON.

Solution.

If tangents be drawn to a conic from two points F_1, F_2 to a conic S , and the whole be reciprocated with respect to a point O , then corresponding to F_1, F_2 we get two lines L_1, L_2 , and to the other intersections of the tangents to S from F_1, F_2 there correspond the other two pairs of straight lines through the intersections of S with L_1, L_2 . In the particular case in which F_1 and F_2 are the circular points at infinity L_1 and L_2 together constitute the point-circle at O so that we obtain the required result.

252. [I. 19. a.] *Every whole number greater than 2 is the shortest side of an infinite number of rational plane triangles, whose sides are integers prime to each other.* ARTEMAS MARTIN.

Solution.

We have to shew that given any integer $a(>2)$ there is an infinite number of pairs of integers $b, c(>a)$ which make $[(b+c)^2 - a^2][a^2 - (b-c)^2]$ a perfect square, a, b, c having no common factor.

1. a odd. Solutions will be given by putting

$$b-c=1, \quad a^2-1=md^2,$$

m (obviously not a perfect square) being taken even. Then it is known that the equation

$$x^2 - my^2 = 1$$

admits of an infinite number of solutions in integers, x being in each case odd. The solution is therefore completed by taking

$$b+c=ax. \quad (x>1).$$

2. a twice an odd integer. In this case we take

$$b-c=4, \quad a^2-16=md^2,$$

so that m is necessarily of one of the forms $4n, 4n+1$.

$$b+c=ax,$$

where x is obtained by solving in integers the equation

$$x^2 - my^2 = 1,$$

which, as before, admits of an infinite number of solutions in each of which x is odd. So that b and c are both odd.

3. a twice an even integer

$$b-c=2, \quad a^2-4=md^2,$$

$b+c=ax$, where x satisfies

$$x^2 - my^2 = 1.$$

271. [J. 1. a.] Find the number of ways of dividing a number n into not more than three parts.

C. F. W. SANDBERG (O).

Solution.

We want to find the number of ways in which we can assign positive integral values (including zero) to r, s, t ($r \geq s \geq t$), so that

$$r + s + t = n.$$

When only two parts are required the number of ways is obviously $1 + \frac{n}{2}$ or $1 + \frac{n-1}{2}$ according as n is even or odd, i.e., it is $I\left(\frac{n+2}{2}\right)$. Thus

when in the case of three parts r has an assigned value $\left[\geq I\left(\frac{n}{3}\right)\right]$, the number of ways of choosing s, t is $I\left(\frac{n-3r+2}{2}\right)$. Therefore the total number of ways of choosing r, s, t is

$$\begin{aligned} & \sum_{r=0}^{I\left(\frac{n}{3}\right)} I\left(\frac{n-3r+2}{2}\right) \\ &= \sum_{r=0}^{I\left(\frac{n}{3}\right)} \left(\frac{n-3r+2}{2}\right) - \frac{1}{2} I\left(\frac{1+I\left(\frac{n}{3}\right)}{2}\right) \text{ if } n \text{ is even,} \\ &= \sum_{r=0}^{I\left(\frac{n}{3}\right)} \left(\frac{n-3r+2}{2}\right) - \frac{1}{2} \left[1 + I\left(\frac{1}{2} I\left(\frac{n}{3}\right)\right)\right] \text{ if } n \text{ is odd,} \\ &= \frac{1}{4} \left[1 + I\left(\frac{n}{3}\right)\right] \left[2n + 4 - 3I\left(\frac{n}{3}\right)\right] - \frac{1}{2} \left[I\left(\frac{n+1}{2}\right) - I\left(\frac{n+1}{3}\right)\right] \end{aligned}$$

whether n be even or odd,

$$= \frac{1}{12} [(n+3-s)(n+4+s) - \overline{n+1} + 3t' - 2s']$$

where s, s', t' are the remainders obtained on dividing $n, n+1, n+1$ by 3, 3, 2 respectively,

$$\begin{aligned} &= \frac{1}{12} [(n^2 + 6n + 11) - s - s^2 - 2s' + 3t'] = 1 + I\left(\frac{n(n+6)}{12}\right) \\ &\because -6 \geq 3t' - s - s^2 - 2s' \geq 1. \end{aligned}$$

287.

All the quantities being real, prove that the limits of

$$\left[\sum_{r=1}^{p-1} x_{qr}^2 - \sum_{r=1}^{p-2} x_{qr} x_{qr+q} \right] / \left[\sum_{r=1}^{pq-1} x_r^2 - \sum_{r=1}^{pq-2} x_r x_{r+q} \right]$$

are 0 and q .

F. S. MACAULAY

Solution by W. E. HARTLEY.

$$\text{We have } \sum_{m=1}^n y_m^2 - \sum_{m=1}^{n-1} y_m y_{m+1} = \frac{1}{2} \left[y_1^2 + y_n^2 + \sum_{m=1}^{n-1} (y_m - y_{m+1})^2 \right]$$

and

$$\left[\sum_{m=1}^n y_m \right]^2 = n \sum_{m=1}^n y_m^2 - \sum_{s=1}^{n-1} \sum_{t=s+1}^n (y_s - y_t)^2.$$

The quantity under consideration is therefore equal to

$$\frac{x_q^2 + x_{p-q}^2 + \sum_{r=1}^{p-2} (x_{qr} - x_{qr+q})^2}{x_1^2 + x_{pq-1}^2 + \sum_{r=1}^{pq-2} (x_r - x_{r+q})^2},$$

and is therefore necessarily positive, [and by making $x_{pr}=0(r=1 \dots \overline{p-1})$ (while the other x 's are finite), takes value 0,]

$$= q \left[x_1^2 + \sum_{r=1}^{p-1} (x_{r+1} - x_r)^2 \right] + q \left[\sum_{r=q(p-1)}^{pq-2} (x_r - x_{r+1})^2 + x_{pq-1}^2 \right] + q \sum_{r=q}^{pq-q-1} (x_r - x_{r+1})^2$$

- sum of squares

$$/ \left[x_1^2 + x_{pq-1}^2 + \sum_{r=1}^{pq-2} (x_r - x_{r+1})^2 \right]$$

$$= q \left[x_1^2 + x_{pq-1}^2 + \sum_{r=1}^{pq-2} (x_r - x_{r+1})^2 \right] - \text{sum of squares}$$

$$/ \left[x_1^2 + x_{pq-1}^2 + \sum_{r=1}^{pq-2} (x_r - x_{r+1})^2 \right]$$

so that the upper limit is q . Hence the limits of the quantity under consideration are 0 and q .

306. [J. 1. a.] A, B , and C have n articles to divide between them subject to the condition that B must not have more articles than either A or C . Show that the number of arrangements is given by the coefficient of x^n in the expansion of $(1-x)^{-2}(1-x^3)^{-1}$. If, however, the condition be that B is not to have fewer than either A or C , show that the generating function is

$$(1+x^3)(1-x)^{-1}(1-x^3)^{-1}(1-x^3)^{-1}.$$

P. A. MACMAHON.

Solution.

We want to find the number of ways in which we can choose positive integers α, β, γ so that $\alpha + \beta + \gamma = n$, $\beta \nless \alpha$, $\beta \nless \gamma$. We may therefore write $\alpha = \beta + a$, $\gamma = \beta + c$, and we have now to find the number of ways of choosing positive integers (including zero) a, β, c such that $a + 3\beta + c = n$. The required number is obviously the coefficient of x^n in the expansion of

$$(1+x+x^2+\dots)(1+x^3+x^6+\dots)(1+x+x^2+\dots),$$

i.e. of $(1-x)^{-2}(1-x^3)^{-1}$.

In the second case let $a, a+b, a+b+c$ be the three parts, a, b, c being all positive integers (including zero), and $(a+b+c)$ being B 's share. We have then to determine the number of ways of assigning positive integral values (zero included) to a, b, c so that $3a+2b+c=n$, each arrangement in which b is not zero being counted twice, as either A or C may have $a+b$. The number is the coefficient in the expansion of

$$(1+x+x^2+\dots)(1+3x^2+3x^4+\dots)(1+x+x^2+\dots) = \frac{1+x^3}{(1-x)(1-x^2)(1-x^3)}.$$

307. [I. 2. b.] The shortest side of a rational plane triangle having integral sides prime to each other cannot be less than 3.

ARTEMAS MARTIN.

Solution.

If a triangle have integral sides the shortest of which is 1, then the other two must be equal. Let each be n : then the area of the triangle is

$$\frac{1}{2}\sqrt{n^2 - \left(\frac{1}{2}\right)^2} = \frac{1}{4}\sqrt{4n^2 - 1},$$

which can never be rational since every odd square is of the form $4p+1$.

If the shortest side be 2, then either

(i) the other two are equal; let each be n : then the area is

$$1 \cdot \sqrt{n^2 - 1},$$

which can never be rational.

Or (ii) the other two differ by 1; let them be $n, n+1$: then the area is

$$\begin{aligned} & \left[\frac{1}{4} \sqrt{\{(b+c)^2 - a^2\} \{a^2 - (b-c)^2\}} \right] \\ & \frac{1}{4} \sqrt{\{(n+2)^2 - (n+1)^2\} \{(n+1)^2 - (n-2)^2\}} \\ & = \frac{1}{4} \sqrt{3(2n+3)(2n-1)} \\ & = \frac{1}{4} \sqrt{4(3n^2 + 3n - 2)} - 1, \end{aligned}$$

which can never be rational.

Thus the shortest side of a rational plane triangle having integral sides cannot be less than 3.

311. [I. 1.] *The number $abcd$ is such that $(ab+cd)^2 = abcd$. What four-figure numbers possess this property?* R. PENDLEBURY.

Solution by W. E. HARTLEY, C. E. YOUNGMAN.

We have $(ab+cd)^2 = abcd$,

$$\begin{aligned} \therefore (ab+cd)(ab+cd-1) &= abcd - ab \\ &= ab \times 99. \end{aligned}$$

$\therefore$ if n denote $ab+cd$

$n(n-1)$ is divisible by 99;

also in order that n^2 may be a four-figure number n must lie between 32 and 99—both limits included. Taking the numbers of the form $11p$ or $11p+1$ in these limits, the only ones which make $n(n-1)$ divisible by 9 as well as 11 are 45, 55 and 99: and these give the numbers

2025, 3025, 9801 (to which we might add 0001).

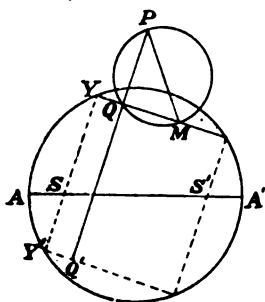
[We see the reasoning above is reversible, so that each of the numbers n which makes $n(n-1)$ divisible by 99 gives one of the four-figure numbers we are seeking, viz. $abcd$ where $n(n-1) = 99 \times ab$, and $cd = n - ab$.]

320. [L¹. 16. b.] (a) *A fixed point P is joined to a point M on a conic, and a circle is described on PM as diameter. Find the envelope of the circles as M moves on the conic, and discuss the cases in which P is (1°) at the centre; (2°) a focus; (3°) a vertex.*

[L¹. 4. c.] (b) *Given two conics C, C' , a tangent to C cuts C' in A and B . Find the locus of the intersections of the tangents to C from A and B .* E. N. BARISIEN.

Solution by C. E. YOUNGMAN.

(a) Draw the tangent at M to the conic C , and on it project P into Q . Then Q lies on the circle PM , and also on the consecutive circle PM' for M'



lies on MQ ; thus the envelope is the locus of Q , or the pedal of C for P . A convenient construction for points QQ' on the curve is:—Take S a focus of C , and $AA'YY'$ the auxiliary circle; draw any chord YY' through S , and a parallel through P , on which project YY' into QQ' . From this follow various properties: the curve has a double point at P , where the tangents are perpendicular to those from P to C ; it has (for central conics) no real points at infinity; it cuts the circle AA' on PS and PS' ; the tangents to it from P are parallel to those from S to the circle AA' , and perpendicular to the asymptotes of C ; and it touches C at the four points whose normals

pass through P . There cannot be five collinear points Q , for then the five lines QM would touch a parabola with focus P , as well as a conic with

focus S . Hence the curve is a quartic, or, when C is a parabola, a cubic. Inverted from P , it becomes a conic—the polar reciprocal of C for the circle of inversion. Hence if C is a circle the curve Q is a limaçon.

In case 1° the curve is a lemniscate; in 2° the auxiliary circle; in 3° the inverse of a parabola—a cardioid when C is a circle, a cissoid when C is a parabola. See Williamson's *Differential Calculus*, chaps. xiv. and xv., Exx.

(b) The locus is a conic touching the common tangents of C and C' . For from Poncelet's Theorem, by first reciprocating the circles into confocal conics, and then projecting these into conics touching four fixed lines, we get: "If a variable polygon circumscribes a conic C , while all its corners but one trace conics of the same four-tangent system with C , then that one also traces another conic of the system." And the case in question is that of a triangle.

327. [B. 2. a.] Show that there exist linear functions u, v of x, y, z which will render

$$u \begin{vmatrix} x^2 & y^2 & z^2 \\ xx' & yy' & zz' \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} + v \begin{vmatrix} x^2 & y^2 & z^2 \\ xx'' & yy'' & zz'' \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \text{ divisible by } \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix};$$

and determine u, v .

W. J. JOHNSTON.

Solution.

Denoting the three determinants by $\Delta_1, \Delta_2, \Delta$ respectively, we see that Δ and Δ_1 vanish when for x, y, z we write x', y', z' , while— α^2, x', x'' , etc. being general— Δ_2 does not vanish. Hence to satisfy the conditions we must have

$$v = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ \lambda & \mu & \nu \end{vmatrix}, \quad \text{and similarly } u = \begin{vmatrix} x & y & z \\ l & m & n \\ x'' & y'' & z'' \end{vmatrix}.$$

But Δ vanishes if $x = x' + \rho x''$, etc. whatever ρ may be; these values of x, y, z give

$$u\Delta_1 + v\Delta_2 = \rho \cdot \begin{vmatrix} xx' & yy' & zz' \\ xx'' & yy'' & zz'' \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} \times \begin{vmatrix} l - \lambda & m - \mu & n - \nu \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix};$$

and this vanishes for all values of ρ if, and only if, $\lambda - l = \alpha x' + \beta x''$, etc. Hence,

$$u = \begin{vmatrix} x & y & z \\ l & m & n \\ x'' & y'' & z'' \end{vmatrix}; \quad v = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ l + \beta x'' & m + \gamma y'' & n + \alpha z'' \end{vmatrix},$$

l, m, n being quite arbitrary. But we notice that the value of u is unaltered by writing for $l, m, n, l + \beta x'',$ etc. so that we may say u, v are always $[x, m, z''], [x, y', n]$, but corresponding to a given value of u, v is indefinite in so far that we may add to it any multiple of Δ .

344. [L. 10. b.] OP, OQ, OR are normals to a parabola, focus S , vertex A . Prove that $\Sigma \hat{A}SP = 2n\pi + 2\hat{A}SO$.

R. W. GENESE.

Solution.

The line $y = mx - 2am - am^3$
is normal to the parabola $y^2 = 4ax$,
and if P be the point at which it is normal $\angle ASP = -2 \tan^{-1} m$.

The values of m which make the normal go through a fixed point O are the roots of the cubic.

$$am^3 + m(2a - x) + y = 0.$$

$$\therefore \Sigma m = 0, \quad \Sigma m_2 m_3 = 2 - \frac{x}{a}, \quad m_1 m_2 m_3 = -\frac{y}{a};$$

m_1, m_2, m_3 being the three roots.

$$\begin{aligned} \therefore \tan \angle ASO &= \frac{y}{a-x} = \frac{-m_1 m_2 m_3}{\Sigma m_2 m_3 - 1} = -\frac{\Sigma m - m_1 m_2 m_3}{1 - \Sigma m_2 m_3} \\ &= -\tan(\Sigma \tan^{-1} m) \\ &= \tan\left(\frac{1}{2} \Sigma \hat{A}SP\right) \end{aligned}$$

$$\therefore \Sigma \hat{A}SP = 2\pi + 2\hat{A}SO.$$

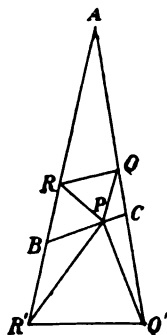
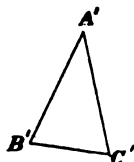
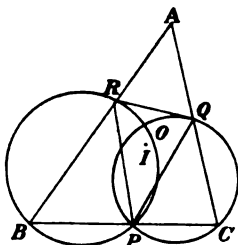
345. [L. 2. c.] Find geometrically the locus of the in-centres of triangles, vertices on three given lines, and equiangular with a given triangle.

J. F. HUDSON.

Solution.

Let ABC be the triangle formed by the three lines; $A'B'C'$ the triangle to which PQR is to be similar; O the point at which BC, CA, AB subtend angles respectively equal to $A + A', B + B', C + C'$. P any point in BC ; Q the intersection of CA with $\odot CPO$; R the intersection of BA with $\odot BPO$. Then

$$\begin{aligned} \angle RPQ &= \angle RPO + \angle OPQ = \angle RBO + \angle OCQ \\ &= \angle BOC - \angle BAC = A'. \end{aligned}$$



Also POR, POQ being supplementary to B, C, QOR is supplementary to A ; and angles Q, R are respectively equal to B' and C' , so that PQR is a triangle satisfying the required conditions (conversely if PQR satisfy conditions, $\odot AQR, BRP, CPQ$ conintersect at O). Since PO, QO, RO make fixed angles with the sides PQR , we have, if $P_0Q_0R_0$ be the pedal triangle of O, I_0 its incentre, I the incentre of PQR ,

$$\begin{aligned} OI_0 : OI &= OP_0 : OP, \text{ and } \angle IOI_0 = \angle POP_0; \\ \therefore \angle OI_0I &= \angle OP_0P \end{aligned}$$

a right angle, so that the locus of I is a line through I_0 perpendicular to OI_0 .

If the triangle be turned over so as to be perversely similar to $A'B'C'$, we get a new locus corresponding to the angles $-A', -B', -C'$ instead of $+A', +B', +C'$.

347. [P. 3. b.] A variable circle, which inverts from each of two fixed points into circles of constant radii, envelopes a pair of circles. C. E. M'VICKER.

Solution

Let A, B be the fixed centres of inversion. Then since
radius of any circle C : radius of the inverse circle C' with respect to A
= tangent to C from A : tangent to C' from A
= sq. of tangent to C from A : sq. of radius of inversion,

and similarly for the other inversion; we have that the tangents to C from A and B are in a constant ratio. Let L be the limiting point of the system of coaxial circles determined by circle C and line AB . Then $AL:BL$ is a fixed ratio. Hence L describes a circle with centre on AB , and L' the inverse of L with respect to A also describes a circle with its centre O on AB . Now a system of coaxial circles inverts with respect to any point on the radical axis into another system with the same radical axis, the limiting points inverting into the limiting points of the new system. Hence the tangent from O to $C' = OL$ is a constant, and the radius of C' is constant. Hence C' envelopes a pair of circles with a common centre O . Hence C also envelopes a pair of circles.

Solution by C. E. YOUNGMAN.

Let A, B be the fixed points, and V, V' two positions of the variable circle, From either A or B , say A , V and V' invert into equal circles O, O' ; $\therefore A$ and B lie on a circle coaxial with V, V' (Casey's *Sequel*, vi. iv. 5); $\therefore$ the inverse of this—a line cutting AB at B' the inverse of B —is coaxial with O, O' (*Ib.* vi. v. 6, Cor. 6); $\therefore$ tangents from B' to O and O' are equal; i.e. the tangent from B' to O is constant. Hence O (being of fixed size) touches two fixed circles with centre B' , and $\therefore V$ touches other two coaxial with A and B .

350. [K. 2. a.] *The triangle formed by the Simson lines of A', B', C' with respect to a triangle ABC is similar to the triangle DEF , where A', B', C' are the mid points of BC, CA, AB respectively, and D, E, F the points of contact of the inscribed circle of ABC .* W. J. G.

Solution by N. QUINT.

This theorem is a particular case of that given by Mr. J. Alison (*Proc. Edin. M. Soc.*, III., p. 85): "if E, F, G be the vertices of any triangle inscribed to a circle ABC , the Simson lines of E, F, G form a triangle similar to EFG "; for as is easily seen, the triangle $A'B'C'$ is equiangular with the triangle DEF formed by joining the points of contact. It may be remarked that—as in many questions dealing with Simson lines—the theorem still holds if we replace the Simson lines by the *general* Simson lines, obtained by taking the oblique projections of A', B', C' . For it has been shown ("The general Wallace line of an in-polygon" (Quint.) *Nieuw Archief voor Wiskunde*, III. p. 153, 1897) that the angle between the general Simson lines for O, O' on the circle for any inscribed triangle is equal to half the arc OO' .

360. [K. 10. e.] *In a semicircle ABA' , centre C , a chord $A'B$ is drawn, and a radius CD parallel to $A'B$. Find the locus of intersection of $CB, A'D$.*

C. BICKERDIKE.

Solution.

Let O be the intersection; $\hat{OCA} = \theta$, $OC = r$, $CA = a$,

then $CO : CB :: CD : CD + A'B$

or $r : a :: a : a + 2a \cos \frac{\theta}{2}$;

$$\therefore r \left(1 + 2 \cos \frac{\theta}{2} \right) = a,$$

the equation to the locus.

362. [K. 10. e.] *If an arc AB of a circle be divided equally in M and unequally in P, find a relation between the cords PA, PB, PM, AM analogous to* *Eucl. II. 5.*

W. J. JOHNSTON.

Solution by PROPOSER.

Let the point P lie between M and B. Take a point Q in the arc AM, such that arc AQ = arc MP. Then arc AM = arc QP, and arc QM = arc PB. Apply Ptolemy's Theorem to the quadrilateral AQMP. Then noticing that the chords of the preceding pairs of equal arcs are equal, we obtain the relation

$$AP \cdot PB + MP^2 = MB^2.$$

363. [L. 3. a.] *Given two points on an ellipse and the positions of its axes, construct the lengths of the axes.*

A. LODGE.

Solution by R. B. WORTHINGTON.

Construction. Let PP' meet axis major CMM' in T. Join PM', P'M, meeting in H. Draw NHR || PM, PM', and meeting the semicircle CRT in R, PP' in Q.

Then shall CR = $\frac{1}{2}$ axis major.

For TP'QP is harmonic, $\therefore$ NQR is the polar of T, both for ellipse and auxiliary $\odot$.

TRC is a right angle; $\therefore$ R is evidently on the auxiliary circle. $\therefore$ CR is the $\frac{1}{2}$ axis major.

Let MP meet auxiliary in p, and take CB : CR = MP : Mp; CB is $\frac{1}{2}$ axis major.

365. [K. 20. d.] *An ass is tied to a peg in the centre of a rectangular plot of grass, sides a, 2a; if he can graze over just half the area, show that the length of his halter must be $\frac{1}{2}a \operatorname{cosec} 59^\circ 4' 51.6''$ very nearly.*

C. E. M'VICKER.

Solution by PROPOSER; R. F. DAVIS; E. M. RADFORD; J. F. HUDSON; and others.

It is easily seen that we are led to the equation

$$2\theta + \sin 2\theta + 2 \cos 2\theta = 2.$$

By experiment $\theta = \frac{\pi}{3}$ nearly, $= \frac{\pi}{3} + x$ (say).

Hence $2\left(\frac{\pi}{3} + x\right) + \sin\left(\frac{2\pi}{3} + 2x\right) + 2 \cos\left(\frac{2\pi}{3} + 2x\right) = 2.$

Expanding the sine and cosine we have

$$x = -\frac{18 - 4\pi - 3\sqrt{3}}{6(2\sqrt{3} - 1)} + \left(\frac{2 - \sqrt{3}}{2\sqrt{3} - 1}\right)x^2 + \frac{2(2\sqrt{3} + 1)}{3(2\sqrt{3} - 1)}x^3 \dots;$$

or, writing β for the first term :

$$x = -\beta + .1087411x^3 + .12077698x^5 \dots,$$

from which by continued approximation

$$\begin{aligned} x &= -\beta + .1087411\beta^3 - .12314190\beta^5 \dots \\ &= -.0180624 + .0000280 - .0000051 \dots \end{aligned}$$

[the term in β^4 will be found to be $< .01$];

$$\begin{aligned} \therefore \theta &= 1.0311580 \dots \text{ radians} \\ &= 59^\circ 4' 51.6''. \end{aligned}$$

[This problem was set (in a modified form) in "Tit-Bits" as a puzzle!]

366. [K. 3. b.] ABC, DBC are two equilateral triangles on the same base BC . A point P is taken on the circle, centre D , radius DB or DC . Show that PA, PB, PC are the sides of a right-angled triangle. E. M. RADFORD.

Solution by C. V. DURELL and others.

AD bisects BC at right angles. $OD = AB \cos 30^\circ$; $BO = \frac{AB}{2}$.

$$\therefore PB^2 + PC^2 = 2PO^2 + 2OB^2 = 2PO^2 + \frac{AB^2}{2},$$

$$PA^2 + PD^2 = 2PO^2 + 2OD^2 = 2PO^2 + \frac{3AB^2}{2}.$$

But $PD^2 = BD^2 = AB^2$; $\therefore$ etc.

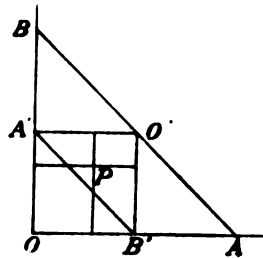
367. [J. 2. c.] A straight line of length unity is divided at random into three parts, subject to the condition that they may form the sides of a triangle. Show that the expectation of the area of the circle inscribed in the triangle is $\frac{\pi}{240}$.

W. ALLEN WHITWORTH.

Solution by C. E. M'VICKER.

Let a, b, c be the parts of the line.

$$\begin{aligned} \text{Area of circle} &= \frac{\pi}{s}(s-a)(s-b)(s-c) \\ &= \frac{\pi}{s}(s-a)(s-b)(a+b-s). \end{aligned}$$



$$\therefore \text{expectation of area} \equiv p = \frac{\pi}{s} \iint (s-a)(s-b)(a+b-s) da db / \iint da db,$$

where $a < s$; $b < s$; $a+b > s$.

To exhibit the limits of integration geometrically, take a representative point P whose coordinates referred to two rectangular axes are a, b ; make $OA = OB = 2s$, the length of the line. It will be seen from the preceding inequalities that P must lie on the triangle $O'A'B'$, whose vertices are the mid-points of the sides of OAB .

Taking $O'A', O'B'$ as new axes of reference,

$$p = \frac{\pi}{s} \iint xy(s-x-y) dx dy / \frac{s^2}{2} = \frac{2\pi}{s^2} \iint xy(s-x-y) dx dy,$$

the integration extending over $O'A'B'$.

$$\therefore p = \frac{2\pi}{s^3} \cdot \frac{s^5}{120} = \frac{\pi s^2}{60} = \frac{\pi}{240} \quad [\because s = \frac{1}{2}]$$

372. [K. 13. a.] A plane figure $ABC\dots$ receives any rigid-body displacement in space to the position $A'B'C'\dots$. Prove that the mid-points of $AA', BB', CC', \dots$ are co-planar, and that the planes bisecting these lines at right angles have a common point. C. E. M'VICKER.

Solution by J. BLAIKIE.

(1) Join $AA', BB', \dots$ and let $A'', B'' \dots$ be the mid-points of the joining lines. Take any point O as origin, and employ the method of Vectors. Since $ABCD$ is congruent with $A'B'C'D'$, if we take

$$AD = mAB + nCD, \text{ we have also } A'D' = mA'B' + nC'D' \dots\dots\dots(1).$$

$$\therefore OD = OA + AD = OA + mAB + nCD.$$

$$OD' = OA' + A'D' = OA' + mA'B' + nC'D'.$$

$$OD'' = \frac{1}{2}(OD + OD') = \frac{1}{2}(OA + OA') + \frac{m}{2}(AB + A'B') + \frac{n}{2}(CD + C'D').$$

$$\begin{aligned}\text{Now } A''B' &= OB' - OA'' = \frac{1}{2}(OB + OB') - \frac{1}{2}(OA + OA') \\ &= \frac{1}{2}(OB - OA) + \frac{1}{2}(OB' - OA') = \frac{1}{2}(AB + A'B'). \\ \therefore OD' &= OA'' + mA''B' + nC''D'';\end{aligned}$$

from which we see that D' is co-planar with A'', B', C'' . Similarly for other points. It is to be observed that this proof holds good when the figures are similar but not equal, and in all cases where equation (1) is satisfied.

(2) Since O is any point, we may assume it to be the point of intersection of the normal planes to AA', BB', CC' , through A'', B'', C'' . Then $OA = OA'$, etc., and the tetrahedron $OABC$ is either congruent with, or is the image of, the tetrahedron $O'A'B'C'$. In the latter case we may take a point O' , the image of O in the plane $A'B'C'$, and the tetrahedrons $OABC, O'A'B'C'$ are congruent. Hence, by superposition, either $OD = OD'$, or $OD = O'D' = OD'$. $\therefore OD'$ is perpendicular to DD' . Similarly for other points.

385. [K. 13. a.] Show that any displacement of a rigid body is equivalent to a series of reflexions in plane mirrors of the points of the body. W. J. JOHNSTON.

Solution by J. BLAIKIE.

Let A, B, C be three points in the body, and let A', B', C' be their positions after displacement. Join AA' , and draw a plane through its mid-point perpendicular to it. Let $A'B'C''$ be the image of ABC in this plane. Join $B'B'$. Then since $A'B' = A'B'$, the plane through the mid-point of $B'B'$ at right angles to it will pass through A' . Let the image of $A'B'C''$ in this plane be $A'B'C'''$. Then it is clear that $A'B'C'$ is the image of $A'B'C'''$ in a plane through $A'B'$ bisecting $C'C'''$ at right angles. If D be a fourth point in the rigid body, a fourth reflexion in the plane $A'B'C'$ will be required to make the final image of D agree with its displaced position D' .

386. [L. 3. c.] If a circle through the centre of an ellipse cut pairs of conjugate diameters in $A, A'; B, B'; \dots$ then shall the chords $AA', BB', \dots$ all pass through a fixed point. A. LODGE.

Solution by W. F. BEARD.

Conjugate diameters from a pencil in involution. Thus the pencil $C(AA'BB' \dots)$ is in involution, and therefore (*vide* Russell's *Pure Geom.*, ch. xx., § 2), $AA', BB', \dots$ are concurrent.

Solution by R. F. DAVIS.

Let CP, CD be conjugate semi-diameters of the ellipse, CD touching the given circle at C . Along the tangent at P take any points Q, Q' such that $QP \cdot PQ' = CD^2$; then CQ, CQ' are conjugate diameters intersecting the given circle in A, A' . Produce CP to U so that $CP \cdot PU = CD^2$.

Then the variable circle $CQ'Q$ passes through a fixed point U . Hence if the constant of inversion from C be chosen so that the tangent at P inverts into the given circle, AA' passes through a fixed point on CP .

This theorem obviously gives by orthogonal projection Frégier's property.

28. [K. 1. c.] Find a point P in the base BC of a triangle ABC , such that $AI^2 - BP \cdot PC$ is a max. or min. Discuss the different cases. E. M. LANGLEY.

Solution by G. HEPPEL.

Draw AN perpendicular to BC . Bisect BC in O . Take Q on BC such that $ON = QN\sqrt{2}$. The cases are distinguished as $AQ \geq BO$.

If $OB = OC = a$; $ON = h$; $AN = k$; $OP = x$;

$$u = AI^2 - BP \cdot PC = k^2 + \frac{1}{2}h^2 - a^2 + \frac{1}{2}(h - 2x)^2 = AQ^2 - BO^2 + \frac{1}{2}(h - 2x)^2.$$

If $u = 0$, $x = \frac{1}{2}h \pm \sqrt{\frac{1}{2}(BO^2 - AQ^2)}$.

If $AQ > BO$; no max. and a min. for $x = \frac{1}{2}h$.

$AQ = BO$; min. is zero.

$AQ < BO$; min. zero for $x = \frac{1}{2}h \pm \sqrt{\frac{1}{2}(BO^2 - AQ^2)}$; max. for $x = \frac{1}{2}h$.

32. [P. 3. b.] *If a system of coaxal circles, real limiting points L, L' , be inverted from any point E in their centre-join, with radius of inversion $\sqrt{EL \cdot EL'}$, then each circle of the system inverts into another circle of the system, and the circle through E inverts into the radical axis of the system, and is the circle of similitude of each pair of mutually inverse circles.* H. D. ELLIS.

Solution by R. F. DAVIS.

Let AOB be the radical axis bisecting LL' at right angles in O . Describe the circle whose centre is O and radius OL , and through E draw any secant EPQ . Let the tangents TP, TQ meet LL' in U and V respectively.

Since $EP \cdot EQ = EL \cdot EL' = \text{square of radius of inversion}$, the point P inverts into the point Q . The circle LPQ is its own inverse; and a circle intersecting orthogonally the circle LPQ at P whose centre lies on LL' inverts into a circle also intersecting orthogonally the circle LPQ at Q whose centre lies on LL' . Hence the circle centre U radius UP inverts into the circle centre V radius VQ : these both belong to the given coaxal system having L, L' as limiting points.

The circle of the system passing through E will invert into the circle of the system having an infinitely large radius, that is, the radical axis AB .

Draw TRM perpendicular to LL' meeting the circle LPQ in R : and join ER which will be the tangent at R . Then since $\{PMQE\} = -1$, $\{UMVE\} = -1$: but E is obviously the external centre of similitude of the circles centres U, V , hence M is the internal centre of similitude of the same circles, and the circle upon EM as diameter their circle of similitude. But the circle upon EM as diameter is the circle of the system passing through E ; for $EM \cdot EO = ER^2 = EP \cdot EQ$.

37. [E. 4. a.] *Find from statical considerations the length of the bisector of an angle of a triangle.* E. M. LANGLEY.

Solution by PROPOSER.

The resultant of two equal forces of bc units along AB, AC respectively is

$$2bc \cos \frac{A}{2}.$$

These forces are $b \cdot AB, c \cdot AC$ respectively, and therefore are equivalent to $(b+c)AD$ along the bisector AD . Hence

$$AD = 2bc \cos \frac{A}{2} / (b+c).$$

39. [A. 1.] *Find a number of two digits equal to the square of the tens' digit together with the square of the sum of its digits.* E. M. LANGLEY.

Solution by PROPOSER.

We have to solve $x^2 + (x+y)^2 = 10x + y$.

The roots of this equation regarded as a quadratic in x are real if $25 - y(y+8)$ is positive, i.e. for $y=0, 1$, or 2 ; further, the roots are rational if $y=0$ or 1 , but irrational if $y=2$. Hence 50 and 41 are the numbers.

42. [K. 5. a.] *I measure a field by pacing. I then make a sketch of the land, using a scale of paces, which is on the scale of 8 inches to a mile. The regulation pace is 32 inches. After drawing my sketch I find that my pace is only 31 inches. What should be the length of a special scale of yards for the sketch to show 1000 yards in all?* J. H. HOOKER.

Solution by PROPOSER.

Length of scale of 1000 yds. for standard pace = $\frac{32}{8} \times 1000 \times 36'' = 4111''$.

Personal pace = $\frac{31}{32}$ (standard pace);

$\therefore$ a distance in sketch reading x yds. is really $\frac{31}{32}x$ yds.;

$\therefore$ scale must be increased in ratio $\frac{32}{31}$;

$\therefore$ length required = $4111'' \times \frac{32}{31} = 4296'' = 119' \cdot 6''$.

Then circle round RDE goes through S, S' , but $\hat{SR}S' = \text{right angle}$;
 $\therefore D, E$ are the points in which the circle on SS' , or on RO_1 as diameter
 meets PP', QQ' ;

$\therefore O_1D, O_1E$ are perpendicular to PP', QQ' .

(i) Thus if O_1 is given fixed point, DE is a fixed straight line.

(ii) If O_2 is the given fixed point.

Then $\hat{DER} = \hat{ERO}_1$; $\therefore RO_1$ is a rectangle

$= \hat{ERO}_2$; $\therefore RO_2O_1$ is pedal triangle of $PP'Q$

$\therefore DE$ is parallel to RO_2 .

Thus the third common tangent is either fixed in (i), or is parallel to a fixed straight line—that joining O to the intersection of the fixed lines—in (ii).

Hence in our original theorem, the fourth point of intersection P is either a fixed point or on a fixed straight line through the point O , and parallel to that joining A and B .

BOOKS, ETC., RECEIVED.

**Differential and Integral Calculus for Beginners.* For students of Physics and Mechanics. By E. EDSEER. pp. vi., 253. 1901. 2/6 (Nelson).

**Non-Euclidean Geometry.* By H. P. MANNING. pp. 95. 1901. 3/6. Ginn and Co. (Ed. Arnold).

Extensions of the Riemann-Roch Theorem in Plane Geometry. By F. S. MAC-AULAY. (Proceedings L. M. S., XXXII., No. 736, pp. 418-430.)

Sur les Triangles Trihomologiques. By J. A. THIRD. pp. 4. (Mathesis, 1900).

Questions de Mécanique. By Drs. X. ANTONARI and C. A. LAIBANT. pp. 224. (Nony.)

Famous Geometrical Theorems. By W. W. RUPERT. Parts I. and II. pp. 1-58. 6d. each. 1900. Heath's Mathematical Monographs (Isbister).

On Teaching Geometry. By FLORENCE MILNER. pp. 18. 6d. 1900. Heath's Mathematical Monographs (Isbister).

**Differential and Integral Calculus.* By E. W. NICHOLS. pp. xi., 394. 1900. Heath and Co. (Isbister).

Le Matematiche Pure ed Applicate. Feb. 1901. Vol. I. No. I. pp. 24. Edited by PROF. C. ALASIA. 10 lire per ann. (Lapi. Città di Castello).

Il Pitagora. Edited by PROF. FAZZARI. Dec.—Jan., 1901. ("Era Nova," Palermo.)

The American Mathematical Monthly. Edited by Prof. B. F. FINKEL, A.M., and J. COLAW, M.A. Feb. and March, 1901.

Gazeta Matematica. Edited by I. IONESCU. Feb., 1901. (Göbl, Bucharest.)

Periodico di Matematica. Edited by PROF. LAZZERI. Anno XVI. Fasc. III. Nov.—Dec., 1900. Anno XVI. Fasc. IV. Jan.—Feb., 1901. And Supplemento. Anno IV. Jan.—Mar., 1901. Fasc. III., IV., V.

Math.-naturwiss. Mitteilungen in Austrag des math.-naturwiss. Vereins in Württemberg. Edited by Drs. SCHMIDT, HAAS, and WÖLFFING. April, 1901.

An Elementary Exposition of Grassmann's Ausdehnungslehre, or Theory of Extension. J. V. COLLINS. Reprinted from *Am. Math. Monthly*. pp. 46. 1901.

* Will be reviewed shortly.

ERRATUM.

Delete solution to 366 [K. 3. b.] p. 42. A correct solution will be found on p. 21.

DUBLIN UNIVERSITY PRESS SERIES.

NEW EDITION (THE FOURTH). 2 Vols., 8vo, price 9s. 6d. each.

THE THEORY OF EQUATIONS.

With an Introduction to the Theory of Binary Algebraic Forms.

By WILLIAM SNOW BURNSIDE, M.A., D.Sc., Fellow of Trinity College, Dublin; and
ARTHUR WILLIAM PANTON, M.A., D.Sc., Fellow and Tutor of Trinity College,
Dublin.

*. To Vol. II. of this new Edition has been added a new chapter on the Theory of
Substitutions and Groups.

LONDON: LONGMANS, GREEN & CO.

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY.

Vol. XVIII., pp. 26, 104,

Session 1899-1900 (7/6). Williams and Norgate.

CONTENTS:—Collignon, Ed., *Note sur un Problème de Géométrie*; Crawford, G. E., *Elementary proof that $\Sigma(a)/n > (\Pi a)^{1/n}$* ; L. Crawford, *The evaluations of a Determinant in the theory of Statistics and Elliptic Geometry*; Dougall, J., *The determination of Green's Function by means of Cylindrical or Spherical Harmonics*; Professor Gibson, *Note on Inequality Theorems; Note on proofs by Projection; Proportion: a Substitute for Euc. V.*; Laisant, C. A., *La section de raison*; Muirhead, R. F., *Dissection of any two triangles into mutually similar pairs; Potential theorems*; Peddie, W., *Elementary proof of Potential Theorems regarding uniform Spherical Shells*; Philip, W. E., *Euler's Theorem on Homogeneous Functions extended; Addition Theorems in Trigonometry*; Russell, A. D., *Dissection of any two triangles, special case.*

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

President :

Professor SIR ROBERT S. BALL, LL.D., F.R.S.

Vice-Presidents :

R. B. HAYWARD, M.A., F.R.S.

J. LARMOR, D.Sc., F.R.S.

R. LEVETT, M.A.

Professor A. LODGE, M.A.

Professor G. M. MINCHIN, M.A., F.R.S.

Treasurer :

F. W. HILL, M.A., City of London School, Victoria Embankment,
London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.

H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

R. F. DAVIS, M.A.

J. M. DYER, M.A.

Professor R. W. GENESE, M.A.

Miss F. HARDCASTLE.

G. HEPPEL, M.A.

Prof. W. H. H. HUDSON, M.A.

F. S. MACAULAY, M.A., D.Sc.

C. E. M'VICKER, M.A.

J. C. PALMER.

Professor H. W. LLOYD TANNER,
M.A., D.Sc., F.R.S.

A. E. WESTERN, M.A.

THE MATHEMATICAL ASSOCIATION, until recently known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all mathematical masters and mistresses, in primary as well as secondary schools, who are interested in promoting good methods of teaching. Such an Association would become a recognized authority in its own department, and would exert an important influence on methods of examination.

General Meetings of the Association will be held in London at least once a term, and in other places if desired. At these Meetings papers on elementary mathematics will be read, and any member will be at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 7s. 6d. per annum, and is due on Jan. 1st. It includes the subscription to the *Gazette*.

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 28.

JULY, 1901.

One Shilling Net.

The Mathematical Association.

THE NEXT GENERAL MEETINGS will be held
at King's College, Strand, London, W.C., at Eight
p.m. on Thursday, October 3, 1901.

The Mathematical Gazette is issued in January, March, May, July,
October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, Alford House, Aberdeen.

All other correspondence should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

NOTICE.—The Index and Title-page to Volume I. can now be obtained from the Publishers.

CONTENTS.

	PAGE
THE TEACHING OF MATHEMATICS. E. M. Langley, M.A. ; C. Godfrey, M.A. ; A. W. Siddons, B.A., ...	105
MATHEMATICAL NOTE: A. C. Dixon, M.A., ...	111
REVIEWS AND NOTICES. Professor G. B. Mathews, F.R.S. ; E. M. Langley, M.A. ; R. F. Davis, M.A., ...	113
SOLUTIONS, ...	124

Will appear shortly :

The Trigonometry of the Tetrahedron. (Rev. G. Richardson, M.A.)

Tripolar Coordinates. (G. N. Bates, M.A.)

Notes on Conics in Areals. (T. J. I'a Bromwich.)

Reviews by Professor E. B. Elliott, F.R.S. ; Professor G. B. Mathews,
F.R.S. ; Professor H. W. Lloyd Tanner, M.A. ; T. J. I'a Bromwich,
M.A. ; W. J. Dobbs, M.A. ; J. H. Grace, M.A. ; E. M. Langley, M.A. ;
W. H. Macaulay, M.A., and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. *No. 8 is out of print.* A few numbers may be obtained from the Editor at the following prices, post free :

No. 7, 4s. ; Nos. 9-18, 1s. each.

The set of 7, 9-18, 14s.

These will soon be out of print and very scarce.

THE MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON :

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

DECEMBER, 1901.

No. 30.

THE TEACHING OF MATHEMATICS.

WHILE all teachers of Mathematics should be interested in the paper read last September by Prof. Perry and in the debate which followed it, members of our own Association, founded as it was for the Improvement of Geometrical Teaching, should give it special attention and should take advantage of the upheaval at Glasgow to press some of their immediate aims. Although Prof. Perry presented his case with a certain amount of picturesque exaggeration, I feel sure that his contentions were in the main true, and that the mathematical training of the schools, however well fitted for those who are being prepared for a University course, turns out the ordinary boy with a modicum of Mathematics which he finds of little use for any professional purpose. I do not however think that we need look upon Prof. Perry as an opponent but rather as a valuable ally. I am convinced that much might be done, at any rate in the earlier stages of mathematical teaching, which besides going far to meet Prof. Perry's views and to remove the reproach (in my opinion deserved) of the Engineer, would be a considerable gain from our own point of view. Our cause may be advanced considerably if we make a proper use of the present opportunity.

The debate at Glasgow ranged widely and there was some vagueness of aim on the part of some of the speakers; but three points emerged with clearness on which there seemed a fair amount of agreement. I will take them under the heads of (i) Examinations (ii) Text-books (iii) Intuitional or Experimental Geometry.

(i) It was admitted that the school system is dominated by examinations, and that little could be done unless Examiners and Inspectors of schools modified the nature of their requirements. A lamentable case of the blighting influence of reactionary examiners was brought forward by Prof. Silvanus Thompson.

(ii) The question of the retention of Euclid as a text-book was again raised. It was urged with considerable force that our retention of a book discarded by other nations had at any rate a presumption against it, and that it was wrong to sacrifice the interests of education to the ease of the examiner.

(iii) More than one speaker pointed out that if the experimental or intuitional method of introducing the truths of Mathematics, and especially of Geometry, were used from the lowest classes of our schools upwards, the strictly deductive course would not lose but gain in effectiveness. I used the greater part of the ten minutes allowed me to urge this point,

and need not enlarge upon it here, though it seems the one thing in which we can all of us work *individually* for improvement without waiting for corporate action. Prof. Perry, in the report of the debate he is about to publish, has kindly allowed me to add some further considerations in favour of the course advocated.

Now the Association seems bound to have something to say in its corporate capacity on each of subjects (i), (ii), and (iii).

As to (i) it might appoint a small committee to keep itself in touch with that appointed by the British Association, and to give the views of the latter, when approved, all the support which, as an organised body of mathematical teachers, it can give, in bringing useful pressure to bear on Examiners and Inspectors.

(ii) It might, on its own account, again approach the Universities and other examining bodies on the subject of text-books, urging, as a sound reason for re-opening the question, the opinion so strongly expressed at Glasgow of the unfitness of Euclid as a text-book for beginners, and asking as first instalment of a much-needed reform, the official recognition of its own work, as a text-book for Examinations. I have always thought that the Council of the A.I.G.T. lost, in 1887, a good opportunity through its liberality of sentiment with regard to other text-books. If it had merely asked that its own should be officially recognised, I fancy it might have succeeded, at any rate at Cambridge; and when no great harm had been found to follow the first step, the authorities would have allowed, by degrees, the greater freedom required. As an old Hon. Sec. it was with great regret that I heard one speaker talk as if the A.I.G.T. had never taken the trouble to bring out a text-book of its own. Such a work not only exists, but has been used with marked success in the Colonies. It was mainly the work of two experienced teachers—Mr. R. B. Hayward of Harrow and Mr. R. Levett of Birmingham—but was circulated in proof among the members of the A.I.G.T. for remark, and only published after minute and searching criticism. It should have received some better recognition than that accorded to it by Oxford and Cambridge, and should have been admitted as a successful attempt—possibly to be followed by other and still more successful attempts—to get an improved text-book of the elements of geometry.

(iii) The Association should formally pass a resolution insisting on the introduction of an intuitional or experimental course of Geometry into all schools to prepare for and accompany the present rigidly deductive course.

I venture to hope that some official action will be taken on these points, and that some of the younger members of the Association with whom the future lies will throw themselves heartily into the work.

EDWARD M. LANGLEY.

THE TEACHING OF MATHEMATICS—A COMPROMISE.

AFTER Professor Perry's stimulating denunciation at Glasgow, many of us must be wondering how far are we really able to mend our ways at public schools? What about Army, Navy, and University examinations? We wish to produce useful engineers, but we must not be too independent; examination requirements are always urgent. Now it may be admitted that the elementary tests imposed by the Universities are quite hopeless. On the other hand, both Navy and Army examiners seem to have been converted by Professor Perry.

Again, how are we to fit the new teaching on to the knowledge which the boy brings to his public school? And what will be the practical details of the arrangement?

I venture to trespass upon your space with a scheme of work which has stood the test of some little experience. We have used it here for the last eighteen months, and are not, I think, dissatisfied with it. It may be described as a compromise; but we hope that Professor Perry, in an indulgent mood, would not condemn it utterly.

SCHEME OF MATHEMATICAL TEACHING.

TERM.	GEOMETRY.	ARITHMETIC.	ALGEBRA.
I.	Experimental work with ruler, compass, protractor, and set squares; paper cutting.	Prime factors, leading to H.C.F. and L.C.M.; fractions; drill in decimal notation; addition and subtraction of decimals; multiplication and division of decimals by single figure; applications of fractions and decimals to concrete quantity; unitary method.	Transition from Arithmetic to Algebra; negative number and its application to quantity; numerical evaluation; use of brackets; drill in notation of Algebra; addition, subtraction, and multiplication, regarded as operations with brackets; positive integral indices; simplest equations and problems.
II.	Euclid I. 1, to I. 26 (omitting 2, 8, 7). Definitions as required. Throughout the Geometry course plenty of easy riders. Geometrical drawing and exercises in measurement to go on side by side with theory.	Multiplication of decimals beginning with left-hand figure of multiplier; division of decimals; decimalisation of money; how recurring decimals occur when we turn a fraction into a decimal; metric system; revision of Term I.	Simple numerical equations in one variable, to be solved in all cases by the use of the four axioms, solutions to be verified always; symbolical expression; problems; easy factors; revision of Term I.
III.	Euclid I. 27—I. 48; part of Book III.	Areas of rectangles, parallelograms, triangles, trapezia, and irregular figures, chiefly by means of squared paper; square and cube root by prime factors; graphical treatment of square root; volumes; simple interest.	Cartesian coordinates; plotting simple graphs; plotting tables of statistics; simultaneous simple equations illustrated by graphs (literal equations to be postponed); problems on simultaneous equations; easy factors and fractions.
IV.	Euclid III., with the exception of Propositions 35, 36, 37.	Rule for square root; how to deal with $\frac{1}{\sqrt{2}-1}$, etc.; contracted multiplication of decimals, with applications (practice; simple and compound interest; involution, etc.).	Factors, leading to H.C.F. and L.C.M.; long division; detached coefficients; method for H.C.F. when factors are not obvious; harder fractions; manipulation of quadratic surds; graphs.
V.	Euclid II., and Propositions 35, 36, 37 of Book III.	Contracted division of decimals, with applications (inverse interest, present worth, discount, etc.); ratio and percentage.	Connections between factors and equations; quadratic equations with problems; reducing degree of equation when one root is obvious.
VI.	Euclid IV., Propositions 1-5, 10, 11, 15. The whole book to be treated as a collection of problems in geometrical drawing.	Revision of contracted multiplication and division; stocks and shares.	Remainder theorem; arithmetic and geometric progressions, without formulae; ratio and proportion.
VII.	Euclid VI., omitting Euclid's treatment of proportion, and substituting an algebraic treatment.	Miscellaneous problems.	Indices, so far as needed to introduce common logarithms; arithmetical problems involving use of logarithm tables; variation; revision of progressions, introducing formulae for same; recurring decimals.
VIII.	Revision of Books III. and VI.	Numerical trigonometry of the acute angle, using sine, cosine, tangent; problems of heights and distances; solution of right-angled triangles, at first by natural sines, etc., afterwards by logarithmic sines, etc.; solution of general triangles by drawing perpendiculars and thus dividing into right-angled triangles. Irrational equations; simple cases of simultaneous quadratics, especially such as can be illustrated by graphs; literal equations; easy permutations and combinations.	

After this stage the following work is taken :

Geometry of the triangle, solid geometry, surfaces and volumes of parallelepipeds, tetrahedra, cylinder, cone, sphere; modern geometry (similar figures, harmonic ranges and pencils, inversion, coaxial circles, pole and polar, cross ratio and involution).

Elementary Trigonometry.

Elementary Dynamics.

Binomial theorem for positive integral index; partial fractions; limits and convergence; general binomial and exponential theorems.

Vectors and complex quantity.

Graphs; asymptotes; slope of a curve; area of a curve.

It should be explained that the amount of time available for this work is, roughly, four hours a week, together with preparation. Many boys would repeat a term's work; but the really bright boy is able to take the work as it is set down, term by term.

CHARLES GODFREY.

THE COLLEGE, WINCHESTER,

Oct., 1901.

FROM A PUBLIC SCHOOL POINT OF VIEW.

THE course of Mathematics suggested by Professor Perry at the British Association "is recommended for training colleges and for boys and girls." It may be admirably adapted to the wants of training colleges, but seems quite impracticable for public schools, though, as we shall see later, there are several suggestions that might well be adopted.

Arithmetic.—"Decimals to be used from the beginning; the fallacy of retaining more figures than are justifiable in calculations involving numbers which represent observed or measured quantities. Contracted and approximate methods of multiplying and dividing numbers whereby all unnecessary figures may be omitted." This is all admirable in principle, but in practice it can be pushed too far; in using approximate methods too soon there is danger of teaching the boy to undervalue accuracy. He says, "this answer is only twopence out," but does not consider the difference between being twopence out in a shilling and twopence out in £100; and again, he does not consider that perhaps his method is wrong, and that this causes the inaccuracy. To take a rather exaggerated example: in finding the true discount on £100 due in a year at $2\frac{1}{2}$ per cent., a boy perhaps finds the interest on £100 for a year, and says that it is less than fifteen pence out. It may be answered that true discount is seldom used, but if this mistake is not corrected, a wrong impression is made on the boy's mind. Professor Perry lays special stress on calculating percentages: surely the first thing is to learn on what the percentage is to be taken. Far and away the most important thing in using approximate methods is to know to what degree of accuracy we are working. This is very difficult for young boys, and, in general, it is better at first to give them carefully chosen examples in which there is no need for approximate work. This enables them to concentrate their thoughts on the methods they are using and the meanings of the technical terms that occur (e.g. interest, discount, quotation, etc., etc.). Then, when they have grasped these, by all means apply approximate methods, and point out that "it is dishonest to use more figures than we are sure of."

"The use of the slide rule" can be taught in a very short time to a boy who has had a fairly good school training; it might perhaps be put in the hands of the older boys, but it would be extremely unwise to give it to a young boy.

"Using algebraical formulae in working examples on ratio and variation." This suggestion is so startling that one could hardly believe it. On referring, however, to Professor Perry's *Practical Mathematics* we find the following: "Compound Proportion.—If 10 labourers dig 150 yards of trench in 5 days of 12 hours, how many labourers will dig 356 yards in 2 days of 9 hours? If

we use l , y , d , and h , we see that the assumption made is

$$l \propto \frac{y}{dh}, \text{ or } l = a \frac{y}{dh}.$$

Hence $10 = a \frac{150}{5 \times 12}$, so that $a = \frac{5 \times 12 \times 10}{150} = 4$.

So that we have the formula $l = 4 \frac{y}{dh}$ for working any exercise. Thus the answer wanted is

$$l = 4 \frac{356}{2 \times 9} = 79.11 \text{ labourers.}''$$

This sort of method shows entire ignorance of the state of the youthful mind. A boy never goes from the general to the particular; in his mind the concrete must always precede the abstract. A young boy learning how to tackle an example involving letters, would first be told to take a *numerical* value for each *letter* in turn, and to consider what he would do with the numbers. He would then be led on to deal in the same way with the letters; in fact, Professor Perry's order would be inverted. What would he say if we followed Professor Cayley, and taught, first of all, geometry of n dimensions, and then descended to the mere particular cases when $n=2$ or 3? This would be quite analogous to the principle proposed above.

There is no mention of the *unitary method*. Judging from the above example in compound proportion, the omission is intentional. There is no better training than the solution of problems by unitary method; its application develops the power of careful thought (as opposed to mere effort of memory) more than anything else in arithmetic. All questions on percentages, interest, present worth, discount, stocks, etc., are best solved by applying this method to the definitions. Professor Perry would treat all these "as the simplest of algebraic exercises."

Algebra.—"Being told in words how to deal arithmetically with a quantity, to be able to state the matter algebraically." This is the method adopted by every teacher in his first lessons in algebra, but surely it is a direct contradiction of the principles involved in the method for compound proportion quoted above. But why are the "Rules of Indices" put before this natural bridge between arithmetic and algebra"? This part of the course is very indefinite. It is difficult to understand exactly what work is included in "Practice in algebraic manipulation generally."

Geometry.—In introducing his scheme Professor Perry remarked, "I think that men who teach demonstrative geometry and orthodox mathematics generally are not only destroying what power to think already exists, but are producing a dislike, a hatred for all kinds of computation, and therefore for all scientific study of nature, and are doing incalculable harm." Yet in the actual scheme he says, "A good teacher will occasionally introduce demonstrative proof as well as measurement," and in the advanced course he suggests "Demonstrative geometry based upon Euclid." This is rather contradictory; perhaps we may conclude from the last quotation that he wishes the subject to be still taught, but wants Euclid to go. The result would be chaos; there would immediately be ten thousand text-books treating the subject in ten thousand different orders (compare the text-books on geometrical conics). Of course they have abandoned Euclid on the Continent and in America, but the circumstances are rather different. In England between 12 and 14 a boy goes from a preparatory school to a public school. This would probably necessitate a change of text-book and consequently a change of order of propositions. For some time he would not understand the master and the master would not understand him. This is not the case in foreign countries, where the schools are largely under State control (the English public schools rejoice—rightly or wrongly—in thinking this a long way off

for them), and so there is more uniformity: and in Germany, for instance, the boy has the same teacher throughout his nine years' course in the Gymnasium, Realgymnasium or Oberrealschule (which to some extent correspond to our Public Schools), and his examination, at the end of his course there, is not for boys of different schools, but is limited to one, so that it does not matter what text-book is used so long as its propositions follow one another in a logical sequence.

Considering the complexity of our educational system in England, it seems to be essential that we should have a standard order for propositions in demonstrative geometry. The only possible chance of any new order being agreed to is its adoption by the great examining bodies, such as the Universities and the Civil Service Commissioners, and the most hopeful of reformers would never hope for that. Why not then be contented with Euclid's order (i.e. in proving any proposition in Euclid no proposition may be assumed that does not precede it*)? Surely this allows enough latitude for reform. The proofs of many propositions might be omitted in a first reading, and some might be omitted altogether; the Third Book (except 35-37) might be taken before the Second; the Eleventh Book (except Prop. 17) might be taken at any time after the first thirty-three propositions of the First Book.

The most marked feature of Professor Perry's scheme is the large amount of practical work he suggests under the headings of *Geometry*, *Mensuration*, and *Use of Squared Paper*.

Some such course is adopted in all continental countries. In America they have a very long one,† and some schools in England are doing work of that character, but it is not general, and there is not enough of it in any of the preparatory and public schools. It would be well for every boy to go through such a course with his mathematical master (the science masters do something of the kind in practical physics, but it cannot, in this case, be interwoven with the mathematics, and so loses much of its value). The majority of the work proposed can be done in an ordinary classroom, each boy being provided with a good set of mathematical instruments (including ruler, compasses, set squares, and protractor) and a pair of scissors. Work in which weighing is necessary must be done in a laboratory (it might perhaps be found necessary for this to be done by the science masters, but in any case it should run hand in hand with the mathematical teaching).

Mathematical masters must feel some diffidence in criticising the details of the work proposed, for though many of them have done practical work in physics, yet as a rule they have not much first-hand experience of how this work is done by boys; nevertheless one may venture to note two points in the scheme. (1) The *Mensuration* work consists very largely of *verification of rules* for areas and volumes; it may be a mere matter of language, but is it not preferable, if we must have rules, that the work should be *invention of rules*? (2) Considering the extraordinary slowness with which boys do practical work of this type, the amount proposed seems rather large for an elementary course (a young boy could hardly be expected to use a planimeter with advantage or bother his head with formulae for the volume and surface of a ring). Possibly the pace would increase when a really good course of simple measurement was started before demonstrative geometry (instead of after, as is too often the case at present).

I need hardly point out that Professor Perry seems to have arranged his scheme purely for the sake of enabling the boy to attain a certain mechanical

* See reports of the Cambridge Mathematical Board and the Oxford Board of Natural Science.

† I have not been able, in the limited time I have had for writing this article, to ascertain when this course begins—certainly demonstrative geometry is not begun till the age of 16 or 17. Further information can be obtained from D. E. Smith's *Teaching of Elementary Mathematics*.

power (which, being mechanical, would soon rust unless constantly used). No thought is given to the development of brain power; in fact such a scheme would deaden all power of thinking. The man it would produce might have a fair power of applying a formula, but would not be able to invent one, and any problem that was not of the stereotyped kind he could never attack with the slightest hope of success.

So far we have chiefly looked at the faults of Professor Perry's scheme. Let us briefly consider a few of his suggestions that could be adopted with advantage.

"The use of rough checks in arithmetical work, especially with regard to the position of the decimal point" might be more generally applied than at present. Too much stress cannot be laid on the value of the use of squared paper for mapping statistics, finding areas, solving equations, etc., etc. There is no doubt that its cost has, in the past, prevented it being used in teaching mathematics, but now that it is quite cheap it should be used freely. The practical geometry course should commence sometime before Euclid is begun, and continue side by side with it. The early introduction of trigonometrical ratios and problems in heights and distances would, with squared paper and experimental geometry, add a very real interest to our mathematical teaching.

No attempt has been made in the above to draw up a scheme for this new work (if we may call it so). There are many practical details to be considered.* It would be of great value if masters would, through the *Mathematical Gazette*, let other masters know the result of any experiments they make in improving the teaching of elementary mathematics.

Professor Perry goes on to give an advanced course. This I have left alone, though it is quite possible to modify the work in our middle and lower forms. Examinations compel us at present to teach on somewhat orthodox lines in higher forms, but if the reform is started where feasible, may it not in time become possible even in higher forms to make our methods more modern?

One noteworthy remark Professor Perry makes:—"A good teacher must understand that no examination made by anyone other than himself can be framed which will properly test the result of his teaching." This principle is already observed in the high schools of Germany. A. W. SIDMONS.

HARROW.

MATHEMATICAL NOTE.

103. [D. 6. b.] *Note on the Logarithmic Series.*

From the exponential series, if u is positive,

$$e^u > 1 + u,$$

hence

$$u > \log(1 + u),$$

and

$$\log \frac{1}{1-v} < \frac{v}{1-v},$$

where $u = \frac{v}{1-v}$, so that v is positive and < 1 .

Again, $e^v < \frac{1}{1-v}$, from the series, so that $\log \frac{1}{1-v} > v$.

Let $1-v = (1-x)^{\frac{1}{n}}$, so that $0 < x < 1$, and $v = 1 - (1-x)^{\frac{1}{n}}$.

* Mr. Hurst has written to *Nature* (Feb. 14th 1901) and Mr. Eggar to the *School World* (Oct. 1901) on the subject.

Thus
$$\frac{1}{n} \log \frac{1}{1-x} < \frac{1-(1-x)^{\frac{1}{n}}}{(1-x)^{\frac{1}{n}}}, \text{ but } > 1-(1-x)^{\frac{1}{n}},$$

or
$$\log \frac{1}{1-x} < n \left[(1-x)^{-\frac{1}{n}} - 1 \right], \text{ but } > n \left[1 - (1-x)^{\frac{1}{n}} \right].$$

Now, by the Binomial Theorem,

$$n \left[(1-x)^{-\frac{1}{n}} - 1 \right] = x + \frac{1+\frac{1}{n}}{2} x^2 + \frac{\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}{2 \cdot 3} x^3 + \dots = A, \text{ say ;}$$

$$n \left[1 - (1-x)^{\frac{1}{n}} \right] = x + \frac{1-\frac{1}{n}}{2} x^2 + \frac{\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)}{2 \cdot 3} x^3 + \dots = B, \text{ say.}$$

Let
$$C = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

Then
$$A > C > B,$$

and
$$A > \log \frac{1}{1-x} > B.$$

But
$$A/B = (1-x)^{-\frac{1}{n}},$$

which can be made as near to 1 as we please by taking n great enough. Hence it must be true that

$$\log \frac{1}{1-x} = C = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots$$

This holds when $0 < x < 1$. Writing x^2 for x we have

$$\log \frac{1}{1-x^2} = x^2 + \frac{1}{2}x^4 + \dots$$

Hence, by subtraction,

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots,$$

which therefore holds when $-1 < x < 1$.*

Again, write $\frac{1}{n}$ for v in the above. Thus

$$\log \frac{n}{n-1} < \frac{1}{n-1} \text{ and } > \frac{1}{n}.$$

Let S_n denote the sum to n terms of the series

$$1 + \left(\frac{1}{2} - \log 2 \right) + \left(\frac{1}{3} - \log \frac{3}{2} \right) + \dots + \left(\frac{1}{n} - \log \frac{n}{n-1} \right) + \dots,$$

*The above proof of the logarithmic series holds only when x is real. We see however that when x is positive the sum of the series $A - C$, which consists of positive terms, tends to the limit zero as n increases. This will therefore also hold when x is complex, and thus in any case when $|x| < 1$,

$$C = \lim_{n \rightarrow \infty} n \left[(1-x)^{-\frac{1}{n}} - 1 \right]$$

$$= \text{a value of } \log \frac{1}{1-x},$$

but which value is to be taken has to be decided by considerations of continuity.

which is absolutely convergent, since its n^{th} term is numerically less than $\frac{1}{n-1} - \frac{1}{n}$. Then

$$\begin{aligned} S_n &= 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \log n \\ &= 2 \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n} \right) - \log n, \end{aligned}$$

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n} - \log 2n.$$

By subtraction,

$$S_{2n} - S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{2n-1} - \frac{1}{2n} - \log 2.$$

But $S_{2n} - S_n$ diminishes without limit as n increases, being in fact numerically $< \frac{1}{2n}$, and therefore

$$\log 2 = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots - \frac{1}{2n} \right).$$

Similarly $\log 3 = 1 + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \frac{1}{7} + \frac{1}{8} - \frac{2}{9} + \dots$,
and so on.

A. C. DIXON.

REVIEWS AND NOTICES.

Die partiellen Differential-Gleichungen der mathematischen Physik. Nach Riemann's Vorlesungen in vierter Auflage neu bearbeitet von HEINRICH WEBER. Zweiter Band. Pp. xii., 528 (Braunschweig: Vieweg u. Sohn, 1901).

After an introductory section devoted to the hypergeometric series, Riemann's P-function, and the ordinary linear differential equation of the second order, Professor Weber deals in succession with the conduction of heat, elasticity, electric vibrations, and hydrodynamics. The English reader will here find himself, for the most part, on familiar ground, and may be excused if he reflects with some satisfaction that in this region at any rate his countrymen have done their full share both in discovery and in exposition. The general plan of the work (now complete) has been carried out with consistency and success, and the result cannot fail to prove very serviceable, not as a treatise on mathematical physics (which it does not profess to be), but rather as a guide to the spirit and methods of those parts of modern analysis which are most important in dealing with physical problems. By an attentive study of this treatise the student of physics will obtain a working knowledge of the Fourier analysis, spherical harmonics, Bessel functions, vector calculus, and the theory of the potential; all with direct reference to definite physical problems. Besides this, the variety of subjects dealt with tends to bring out the analogy and correspondence between different branches of applied mathematics which so often proves helpfully suggestive. Finally, it is in a certain sense advantageous to have the fundamental facts and assumptions of mathematical physics laid down in a clear-cut analytical form: this enables us to realise precisely the value of our analysis as an interpretation of Nature, and rightly used, is of service to the experimentalist, by suggesting definite lines of research, and by saving him misdirected and unprofitable labour.

The sections of this volume which will probably be found most interesting by experts are those on electric vibrations and on the motion of a gas. In the first of these Professor Weber discusses Maxwell's equations, and investigates in considerable detail the propagation of current along a wire and the electric oscillations of a spherical conductor surrounded by a dielectric. In the section on the motion of a gas we have an exposition of Riemann's theory of discontinuous motion, which, it will be remembered, has been criticised by Lord Rayleigh as being inconsistent with the principle of energy. Professor Weber's answer to

the objection is that Riemann's analysis leads to a result which, in its general form, implies the existence of impulsive waves of condensation and rarefaction; it is only waves of the latter kind which violate the principle of energy, so that Riemann's investigation appears to be unobjectionable so far as impulsive condensation is concerned. The case is somewhat analogous to problems in potential, etc., where a differential equation supplies us with two independent functions, one of which has to be rejected from physical considerations.

G. B. MATHEWS.

The *Proceedings of the Edinburgh Mathematical Society* (Session 1900-1901). Vol. XIX. contains fifteen papers by various members of the Society. The most interesting is a note by Mr. David Mair, who proves in a very simple way that the n th root of a prime number cannot be the root of an equation of degree less than n with rational coefficients. Attention may be called to Mr. Carslaw's paper on the oblique incidence of a train of plane waves on a semi-infinite plane.

G. B. MATHEWS.

The Teaching of Mathematics in the Higher Schools of Prussia. By J. W. A. YOUNG. Pp. xiv., 141. 2s. 6d. net. 1901. (Longmans.)

The title of this little book should ensure it a good circulation among our Association, founded as it was for the improvement of geometrical teaching, and enlarged to the wider scope indicated by its present name. But while mathematical teaching is the principal object of Dr. Young's investigations, it is by no means the only one. The general status of the higher schools, their relation to the Government, the position and emoluments of teachers, and their influence on mathematical as on other teaching, examinations, are all dealt with thoroughly. It should be studied carefully by all who profess to have any opinions on education, and especially by the governors of our public schools.

Those familiar only with the traditional nature of the appointments in English schools would find matter for reflection in the fact that it has been found possible to secure marked and peculiar success by methods entirely different. They will not, perhaps, be surprised to find that in a nation so distinguished for scientific discipline as the Germans, it is considered necessary for a man to prepare himself seriously for his life's work. Indeed there have been those in England, and some head masters among them, who have thought that preparation and training were as necessary to a schoolmaster as to a dentist. But it will possibly come as a surprise to many to find that in Germany teachers in secondary schools are valued civil servants; that while their salaries are moderate they are relatively considerable, being on the same scale as those for judges; that marked deference is paid to age and experience, responsible posts being usually allotted to elderly men (the market value of a German teacher increasing as he gets older!); that security of tenure seems to produce none of the bad effects that have been predicted for it; and that there is a State pension for every teacher on his retirement. The last two considerations may well make teachers in English schools envy the position of their German brethren, who can work steadily in their profession and give a life's devotion to it, secure that they will not be turned off at forty to get their living by tomato-growing. Turning to the more immediate subject of investigation, full particulars as to the hours allotted to Mathematics in various kinds of schools, and in their various classes, are supplied in a tabular form. The specimens of final examination papers given seem to be of a searching character, and suited, not only for testing a candidate's acquaintance with theory, but his power of applying theory to numerical calculation. Some model lessons are also given; they are well worth the study of those who have to teach the elements of any mathematical subject, but do not appear to differ from those which painstaking teachers in this country, who have taken the trouble to learn their craft, have been in the habit of giving. Few teachers, we hope, nowadays, think that ordinary boys can learn Mathematics by reading text-books or by the mere setting and marking of examples. On the other hand we share the doubts, which Dr. Young seems to have, as to whether the opposite plan of depending simply on oral instruction is not carried to too great lengths in German schools, and we are inclined to think that a boy should, by the age of 14 or 15, be getting into the habit of reading a text-book intelligently, and studying on his own account. In conclusion, we wish the book the wide circulation which its subject deserves and its price renders possible.

E. M. LANGLEY.

Geometrical Exercises from Nixon's "Euclid Revised," with Solutions.
By A. LARMOR, M.A. Pp. vi., 170. 3s. 6d. 1901. (Clarendon Press.)

From its first appearance Nixon's *Euclid Revised* has held a place of honour on the mathematical bookshelf. Not only did it present certain very novel and striking features which were entirely due to its talented author, but it embodied to a large extent the revolutionary ideas of the A.I.G.T.; it discarded the "Pons Asinorum," suppressed useless propositions, and generally shed both sweetness and light upon the dust-encrusted text. To those who, more or less unwisely, neglected to look at their Euclid until the night before an examination it proved a veritable boon, for by its aid the foolish virgin could run through the whole subject in a few hours. The presentation of the various proofs in a kind of "concentrated tabloid" form proved most attractive. Its various Addenda contained full demonstrations of those standard theorems which are intimately connected with the text—then only to be found in M'Dowell's *Exercises* and in Casey's *Sequel*. In addition to this, about nine hundred unsolved riders were proposed, forming on the whole a very complete geometrical armoury.

The present volume contains what we may be permitted to call "miniature" solutions of these riders. We learn from the preface that they were undertaken at the request of Mr. Nixon, who placed at the disposal of the author a large collection of material which he himself had accumulated towards this object before he was overtaken by ill-health. The *raison d'être* of the book is very well set forth:—"In Geometry more independent initiative is demanded, while the directness of the argument and the compactness and elegance of the solution provide a vigorous mental stimulus."

Of the compactness and elegance of the solutions here set forth there can be no possible manner of doubt. Perhaps in some cases the beginner might justifiably ask for a little more light, although in the latter portions the proofs are relatively fuller. Enunciations are given in large type, so that the volume is complete in itself. But in all cases the reader *est prié de faire la figure*. After the beginner has proceeded some little way in Euclid's text and is confronted with some easy theorem of this sort (say)—"the joins of the mid points of the sides of a quadrilateral form a parallelogram"—he has to exercise for the first time a certain intuition as to the shortest and most effective plan of campaign. Later, after much practice and after many failures, this power of intuition becomes much strengthened, and the student finally attains what Jacobi called "glückliche Divination." But, where and when the path he is treading seems to lead over difficult or impassable ground, he must have recourse to the services of a trusty guide to assist, warn, and instruct him on his way. He must grasp the fact that the work done by Euclid two hundred years B.C. is to-day as rigid and useful as ever it was; and also that if on the one hand there is no "royal road" in Geometry, on the other there are no steps to be retraced. One important stimulus in Geometry is the discovery of the value of transformations. Take, for instance, the Similitude theorem:—"Given two triangles whose sides are parallel, then the joins of corresponding points are collinear"—which is perfectly clear from the Sixth Book. Reciprocate it, and it leads to the fundamental theorem of Perspective. Project it, and it leads to the conception of parallel straight lines meeting on the line at infinity.

But to proceed to details.

No. 22, p. 23, and No. 30, p. 24.—If AXB be a right angle, and X either the mid point of AB or the foot of the perpendicular from X on AB , we have in both cases the relation $AP \cdot PB = XP^2$. It therefore follows that if $ABCD$ be a cyclic quadrilateral whose diagonals intersect in X , then the mid points of the sides and the feet of the perpendiculars from X on the sides are eight points all lying on the circular locus $OP^2 + XP^2 = OA^2$, where O is the centre of the circle $ABCD$. Also the quadrilateral formed by joining these feet is noticeable as possibly the simplest instance of a quadrilateral circumscribed to one circle and inscribed to another.

No. 54, p. 69.—It is unnecessary to employ three cyclic quadrilaterals. By construction $ABXQ$ is cyclic; hence $B\hat{X}P = A = B\hat{O}P$, and $BCXP$ is also cyclic.

No. 52, p. 152.—Another method of constructing a triangle of given species with its vertices on three given concentric circles may be noticed. Let ABC be any triangle of the given species; r_1, r_2, r_3 , the radii of the circles centre O . Describe a triangle OaP , having its sides $Oa = r_1$, $aP = (b/a)r_2$, $PO = (c/a)r_3$. Upon either

side of OP describe a triangle $PO\gamma$, similar to ABC , and upon $\alpha\gamma$ describe a triangle $\alpha\beta\gamma$ directly similar to $PO\gamma$. Then $\alpha\beta\gamma$ is the triangle required.

In the solution of No. 74, p. 11, it is shown that the theorem that "bisectors of the internal angles of a quadrilateral form a quadrilateral whose opposite angles are supplementary" is not always true. Beginners are always apt to draw their angles acute-angled and their quadrilaterals equilateral, leaving some vaguely apprehended general principle of continuity to do the rest. But the teacher should always insist on a detailed examination of the possibilities of the figure. Take the simplest instance: if the angle $APB = AQB$, then $APQB$ is cyclic only in the case when P, Q are on the same side of AB . Quite recently our Editor has elsewhere drawn attention to the fact that if the external bisectors of two angles of a triangle are equal, it does not necessarily follow that the triangle is isosceles.

In conclusion, we may make special mention of the elegance of the solutions of Nos. 31, p. 58; 67, p. 133; 109, p. 167; 59, p. 155. All solutions involving inversions are very satisfactorily done, and will repay careful study. We have also read with pleasure those relating to the contact of circles.

To pp. 111-119 attaches a somewhat melancholy interest. In the middle eighties keen enthusiasm was evoked by the New Geometry of the Triangle. Abroad Lemoine, Brocard, d'Ocagne, Neuberg, Vigarié, and at home Tucker, Taylor, Casey, and Simmons, vied with one another in developing new properties. Has all this interesting work failed to make any permanent mark on the Geometry of to-day? Perhaps the swing of the pendulum may again bring in something more interesting to geometers than the factorisation of high integers. Let us hope that an attractive and informing work like this volume of solutions may prove the starting point for fresh geometrical development, by giving a stimulus to some dormant geometrical instinct.

R. F. DAVIS.

(1) **Premiers Principes de Géométrie Moderne.** By E. DUFOURCO. Pp. vii., 160. 3 fr. 1899. (Gauthier-Villars.)

(2) **The "Junior" Euclid, Books III. and IV.** By S. W. FINN. Pp. vi., 144-344. 2s. 1901. (Clarendon Press.)

(3) **Plane and Solid Geometry.** By A. SCHULTZE and F. L. SEVENOAK. Pp. xii., 370. 6s. 1901. (Macmillan.)

(4) **Elementary Geometry, Plane and Solid.** By THOMAS F. HOLGATE. Pp. xii., 440. 6s. 1901. (Macmillan.)

(5) **Famous Geometrical Theorems and Problems, with their History.** By W. W. RUPERT. Four Parts. Parts I. and II. Pp. 58. Ten cents each part. 1900. (Heath, Isbister.)

(6) **On Teaching Geometry.** By FLORENCE MILNER. Pp. 18. Ten cents. 1901. (Heath, Boston.)

(7) **Euclid I.-IV., VI., XI.** By C. SMITH and SOPHIE BRYANT. Pp. viii., 460. 4s. 6d. 1901. (Macmillan.)

(8) **Euclid I. and II.** By T. VARLEY. Pp. iv., 151. Second Edition. 1s. 1901. (Allman.)

(9) **Deductions in Euclid.** By T. W. EDMONDSON. Pp. 206. 2s. 6d. 1901. (Clive.)

(10) **Éléments de Géométrie.** Par F. J. Pp. xii., 523. 3 fr. 60 c. 1900. (Poussielgue.)

(11) **Leçons de Géométrie Élémentaire.** Vol. II. (Géométrie dans l'espace.) By J. HADAMARD. Pp. xxii., 582. 10 fr. 1901. (Armand Colin.)

(12) **Essai sur les fondements de la Géométrie.** By B. A. W. RUSSELL (translated by A. CADENAT). Pp. x., 274. 1901. (Gauthier-Villars.)

(13) **Les Principes Fondamentaux de la Géométrie.** By D. HILBERT (translated by L. LAUGEL). Pp. 114. 1900. (Gauthier-Villars.)

(14) **Traité de Géométrie Descriptive.** By J. DE LA GOURNERIE. In 4. In three parts, each with Atlas, 1891, 1890, 1901. Editions, 3rd, 2nd, and 3rd respectively. 10 frs. each part. (Gauthier-Villars.)

(1) In his introduction the author laments the tendency of the ordinary official course to extend the study of analytical geometry at the expense of Geometry proper. "It seems to me that the study of the works of Poncelet, Chasles, and Laguerre would contribute more than an examination of the different methods of discussing the equation in S to develop the geometrical sense, which after all is the principal aim of the study of Mathematics." (We cannot follow M. Duporcq all the way in the last statement.) He has therefore set himself the task of making the geometric point of view as attractive as possible to the student, forced by the iron laws of a State "programme" to spend the greater part of his time in the uncongenial regions of analysis. This task he has accomplished with considerable success.

The wonderful fertility of modern geometry is due to the introduction of the idea of transformation, by means of which, from a single proposition, we can deduce a number of propositions distinct from the original. These transformations, moreover, may be successive. Double reciprocation is common enough. The Rev. T. C. Simmons developed the properties of harmonic polygons by reciprocations followed by orthogonal projections. Mr. C. E. M'Vicker has in these pages combined inversions with subsequent reflexions.

Poncelet's principle of continuity, extending the generality of algebraical language to geometry, serves the author as a *point de départ*. He then gives, in its most general form, the idea of transformation, throwing into bold relief the notion of contact transformation. The simplest transformations are those which transform points into points and lines into lines, and are respectively called *punctuel* and homographic. After dealing with homographic divisions, pencils, etc., M. Duporcq proceeds to dwell on those transformations in which to a point there corresponds a line, and not another point. The simplest of these *dualistic* transformations is that in which to collinear points correspond concurrent lines. This doctrine of *correlation* naturally follows that of homographic transformation. Now, since homography leaves undisturbed the order of an algebraic curve or surface, and correlation transforms order into class, we see that any application of either to a conic or a quadric will give us properties connected with other conics or quadrics. The value, therefore, of these transformations in the theory of curves and surfaces is obvious. The last chapter deals with various other geometrical transformations, and in particular with inversions—"transformation par rayons vecteurs réciproques," and on those quadratic transformations of which inversion is only a particular case. The author then introduces us to Lie's transformations, in which lines are replaced by spheres. So much for the doctrine. As for the applications, the fourth and fifth chapters present in a really elegant and concise manner the principal properties of conics and quadrics: the conic, the range or pencil of conics, conics harmonically circumscribed or apolar to a conic, and so on. As a study in logical order, these chapters should make a marked impression on the mind of the student.

Those who have read Miss Scott's articles in recent numbers of the *Gazette* will hardly accept the following statement: "Le point . . . ne représente rien de géométrique, lorsqu'il est imaginaire, pas plus qu'une surface à coefficients imaginaires, le tétraèdre de référence étant supposé réel." This is enough to make von Staudt turn in his grave! There is a misprint of (80) for (86) in the note at the foot of p. 107. We notice that the theory of inversion is ascribed to Bellavitis (1836). Bellavitis was the first to make a general statement of the method, but the principle of inversion is traced as far back as Ptolemy by Chasles (*Rapport Hist.*, pp. 140-2), and according to Dr. C. Taylor, Quetelet was acquainted with it in 1827. Müller's German-French *Mathematisches Vocabularium* gives Cramer, 1750, and Magnus, 1832. Who was this Magnus? Can it be the Professor of Chemistry at Berlin (1802-1870)? No "Magnus" appears in the histories of Rouse Ball or Fink. But a "Magnus" wrote on *Die Abweichung der Geschosse* in 1852. If they are the same, he must have been a very Helmholtz for versatility.

(2) Mr. Finn's edition of *Eucl. III.*, IV. is on the lines of the corresponding edition of Books I., II. He draws special attention to his treatment of the identical equality of two triangles, which consists in having a separate picture of the triangles, the sides and angles bearing characteristic marks denoting the three equalities which enable the student to apply the results of *Eucl. I.* 4, 8, 26.

In most of the figures further assistance is given to the student by drawing the lines of the data thicker than those of the construction. The Geometry of the Triangle is not neglected, and on p. 303 is a figure rarely to be found in our text-books—the nine point circle in contact with the in- and ex-circles. There is no mention of Wallace's claim to be considered the discoverer of the pedal line, the theorem in connection with which "is attributed to Robert Simpson." Inversion is just touched on in two pages, but poles and polars, radical axes, coaxial circles, etc., receive fuller treatment. The introduction to Book IV. is likely to prove useful, and the sections on Maxima and Minima and the doctrine of limits will be found adequate for the purposes of the pupil at this stage of his work. In addition to the various riders appended to the propositions there is a collection of easy and carefully graduated riders from Cambridge and Oxford local papers and other sources.

Mr. Finn has not hesitated, where experience has shown it desirable, to divide into parts with separate figures propositions such as III. 7, 8, 31, etc., and he has also included in the text the converse of III. 21, 22, 32, 35, 36. The young teacher will find much that is useful to him in the exercise of his craft in the pages of this edition of Euclid's Elements.

(3) In general plan and order of treatment the work of Messrs. Schultze and Sevenoak does not seem to differ essentially from other American Geometries. But it is especially remarkable for the deliberate attempt that is made by the authors to induce the student to think for himself. For instance, in the propositions the whole of the demonstration is not necessarily given, a hint as to the line of proof to be taken being often considered just enough to set the pupil on his road to the goal. Thus, in a very systematic manner the student is introduced into the mysteries of the "rider." Algebraical language is used without stint in dealing with proportion, and of course in the sections on Arithmetical Geometry. In the section on Regular Polygons the authors show that the circumference of a circle is less than the perimeter of any enveloping line, and that too by a method, the validity of which is not obvious. About 135 pages are devoted to solid and elementary spherical geometry. The diagrams in this division of the book are the best we have seen in any text-book. The prefatory hints to teachers contain several suggestions that may be found useful to those whose experience is small. The difficulty felt by beginners in remembering the data and the hypothesis is met by the use of coloured crayons and other graphic devices.

We have noticed a few slips, such as the following: In the list of symbols and abbreviations that for "is greater than" is wrong; and the small "s" inside $\angle$, $\triangle$, or $\square$ is trying to the eyes and more difficult to make when writing fast than if attached outside. A Scholium is defined as a remark, but §§ 81, 82 are headed respectively by the two terms as if a Remark were not quite the same thing as a Scholium. A circle is defined as a portion of a plane, etc., and we are asked to circumscribe a circle about a triangle. The solutions of the quadratic, p. 222, line 2, is incorrect; in the 2nd, 3rd, 5th, and 6th lines b and a should be interchanged, and the "construction" suffers accordingly.

(4) Professor Holgate covers exactly the same ground as Messrs. Schultze and Sevenoak, but treats the subject rather more fully. After the introductory chapter the pupil familiarises himself with the use of the ruler and compasses, and is thus "introduced to the idea of a formal proof in connection with matters which he clearly sees need proving." As in every day life the circle is never thought of as a portion of a plane; it is considered throughout the book as a particular kind of line, just as the polygon is a figure made up of points and lines. The aim of the authors of these books is to cultivate the geometric sense and the acquisition of geometric knowledge. To these aims logical reasoning and rhetorical demonstration are held as quite secondary objects. The exercises are in all cases well selected, and are intended to be worked through in the order in which they come, that is to say, they are an integral part of the text. Here in England it is difficult to say whether a course of Geometry, such as is here indicated, attains the result desired. It should be worth while to have an experimental class in some of our larger schools, but it should be in charge of a young enthusiast who is not enamoured of Euclidean methods. There is no doubt that a very much larger amount of geometric knowledge can be acquired

in a very much shorter time; and in the case of boys who leave school at an early age, the advantage of having worked through a book covering this ground in the time they usually take to "learn" four books of Euclid, is obvious. The essential question is, will the pupil have the geometric sense in a higher stage of development after such a course than if he had taken up Euclid, or will he assert with vicious warmth that Euclid or Schultze was a "beast who wrote books for the third and fourth forms"?

(5) We have received from Messrs. Isbister the first two parts of Mr. Rupert's monograph. They contain five proofs that the sum of the three angles of every plane triangle are together equal to two right angles, 26 proofs of *Euc. I. 47*, and a chapter on the value of π , half of which is taken up with extracts from *De Morgan's Budget of Paradoxes*.

(6) The aim of Miss Milner's well-meaning little pamphlet may be described as the apotheosis of the syllogism. If this (the syllogistic) "method be used in conventional and formal demonstration, it becomes of the greatest value in original work." With this bold statement we cannot entirely agree, and, in addition, we might observe that the method has the disadvantage of giving the average student a distaste for the subject in its elementary stages.

(7) We have looked forward with much interest to the completion of this School edition of Euclid, partly because the authors are distinguished members of what was once the A.I.G.T. and partly because each has undeniable claims on the respect and attention of mathematical teachers in this country. *Place aux dames!* Mrs. Bryant ranks with another member of our Association—Miss Dorothea Beale—as the most prominent of our head-mistresses, but she has made a reputation in more fields than one. She is a logician as well as a mathematician, a leader of no mean repute in ethics as well as in pedagogy. As for the Master of Sidney, is not his name a household word as the author of works on Arithmetic, Algebra, and Conics? From such collaboration we have therefore a right to expect something of the best, and we are not disappointed. How then is this edition distinguished from the multitudinous rivals in the market at the present moment? To Euclid's "order and manner" the authors rigorously adhere, but are animated by no spirit of slavish devotion to the doctrine of verbal inspiration, which they lament has not yet disappeared from some of our schools. And as for the pupil, it should be "considered distinctly meritorious to depart from the exact words of text-book or teacher, provided that a sound proof is given." With regard to the question of references, raised in vain in the *Gazette* by Mr. Siddons (*No. 27*, p. 58), the authors point out "that these references are given for the convenience of the learner, to show him where to look for some knowledge he may have forgotten. Examiners stand in no need of such help, and by no means wish to impose on the memory of the students the heavy and useless burden of learning how to give accurately numbered references. . . ." Remarks such as these serve to show the eminently sane manner in which the authors approach their subject.

In the First Book we find that no alternative proofs are given for Props. v. and vi., because "experience . . . appears to show that the average beginner finds Euclid's proofs easier to understand—or at any rate easier to reproduce—than the alternatives that have been suggested." We wonder if there is really any solid ground for this contention. It would seem impossible that anything could be simpler than the proof by superposition. As far as the experience of examiners goes, it may well be that the weaker vessels have been invariably taught what seemed to their teachers the easier method of proof, and that after the manner of their kind they distinguished themselves under the stress of the examination room. We note that the ordinary "proof" of Prop. xxiv. has disappeared, and that in Prop. xxii. it is shown that the circles must intersect. The common ambiguity in the use of the term "circle" is pointed out. Simson's enunciations of xviii. and xix. are rejected, useful notes are added to Props. xxxv., xlv., xlvii., xlviii. To Prop. xii. is added a proof that all right angles are equal; Prop. B. is a necessary addition to Prop. xxvi.; and "to describe a triangle equal to a given rectilineal figure" is given with Prop. xiv. from Thomas Simpson's *Geometry*. There is a good collection of worked-out theorems and problems, including a nice group of fifteen properties connected with squares described on the sides of any triangle. That the rectangle contained by the sum and difference of two straight

lines is equal to the difference of the squares on the lines is shown in the note to II. vi., and independently, after Props. ix. and x. To produce a given straight line so that the rectangle contained by the whole line thus produced and the part produced may be equal to a given square, is also given a place in Book II. Lardner's proofs of II. xii., xiii., as an extension of I. xlvii., are wisely given as alternatives. They are to be found in no other modern text-book, and the authors rightly claim that they show directly the equality of the figures and are more instructive than Euclid's proofs. There is a set of seventeen easy exercises on the figure of II. xi. The treatment of Book II. is most successful, but we think that it might have been left to the teacher to decide when a boy is to be permitted to write AB^2 for the sq. on AB , or $AB \cdot CD$ for the rect. AB, CD .

In Book III. the more notable changes are the proofs of Props. ix. and x., and the departure from the Euclidean method of treating the contact of circles. Props. xxvi.-xxix. are proved by superposition. Adequate attention is drawn to the relation between properties of tangents and properties of secants, and to the properties of intersecting and touching circles. In Book IV. we may notice the notes on Prop. x. as very instructive.

Under the title "Theory of Proportion" we have Euclid's test applied to six cases. Book VI. is followed by the usual collection of theorems in similitude, maxima and minima, inversion, etc. At the end of Book XI. are sections dealing with the more elementary properties of the sphere and tetrahedron. More examples might have been set on the "isosceles" tetrahedron, which lends itself very readily to investigation, and which we have found students delight in for their first and innocent experiments in original research. The authors have done well in making the pupil familiar with the names of geometers. There are a few lines on Euclid and his work, and we could wish that space had been found for more in the shape of biographical notes. We extend to the authors our hearty congratulations on the addition of a valuable and instructive text-book to the mathematical literature of our schools, an addition which we feel sure will attain a deservedly large measure of success.

(8) Mr. Varley's Euclid I. and II. has reached a second edition. Evidently some teachers find that beginners get on better when confronted with:—General enunciation, Specific ditto, (a) what is given, (b) what we want to do, or prove, how to do it or prove it. Where alternative proofs are given, the older proofs are in the text or in the appendix. Axiom xii. becomes Prop. 28A. The exercises and riders are carefully graduated, and a few additional propositions are worked out in full. A well-thought-out book for the beginner.

(9) Mr. Edmondson's book of deductions from the first four books of Euclid is evidently meant for the private student who takes such an examination as the London Matriculation. It is neatly and clearly printed, and most of the diagrams are very well drawn. That on p. 17 is incorrect, for CD is not the shortest side. The average reader will, we should think, find many of the demonstrations proved in even too much detail. But for the class for whom such a book is necessary this is erring on the right side.

(10) The *Frères des Écoles Chrétiennes* have well deserved their success as teachers, and French educational literature would be the poorer if the publications of this body were withdrawn from circulation. Of those publications the best are those devoted to Mathematics, and of these again, the best are the book under notice, and its complement, the corresponding *Livre du Maître*, entitled *Exercices de Géométrie*. The "Éléments" covers the ordinary ground of a French Geometry: Plane and Solid Geometry, and the "usual curves," i.e. the parabola, ellipse, hyperbola, and the helix. But the appendix of some two hundred pages takes the student much wider afield. In addition to poles and polars, inversion, directly and inversely similar figures, and the like, he is introduced to the modern geometry of the triangle, to surfaces of the second degree, to the helicoid, spirals, the cycloid and epicycloid, and the catenary. Guldin's theorems, areas and volumes, rules such as Simpson's rule, are discussed and applied to cases such as arise in the construction of bridges, tunnels, canals, etc. About a thousand theorems and problems are worked out in full, and as many more are set for solution. And all this for 3 fr. 50 c. The teacher who has the *Livre du Maître* will have therein the full solutions of over 2500 questions, of which nearly

300 are on the modern geometry of the triangle. As a mine of questions this collection will be difficult to beat. Both books are in thick boards, and will stand hard wear.

(11) This is the sequel to the *Leçons de Géométrie Plane* published in the course of elementary text-books edited by M. Darboux, the well-known Dean of the *Faculté des Sciences de Paris*, whose name needs no commendation to English mathematicians. And this volume is worthy of its place in a series which numbers among its contributors men like Messrs. Jules Tannery, Tisserand, and Andoyer. Although written expressly for students who propose to take the "baccalaureat," it does not merely confine itself to that programme, but may be taken as a complete exposition of elementary modern geometry. To this statement there is but one exception, the author does not deal with the conception of imaginaries. Among the novelties of treatment we notice a very pretty proof of Euler's theorem on polyhedra, in which, by the introduction of the idea of "order of connection," the author claims to have obtained the simplest existing proof, *du moins parmi les démonstrations correctes (on sait qu'il en existe beaucoup d'autres)*. He also ingeniously applies the theory of groups to the proof of the existence of regular polyhedra, introducing in particular "l'importante notion de domaine fondamentale." A remarkable note on the solubility of problems in Geometry is based on the researches of MM. Giacomini, Castelnovo, and Enriques, recently published under the title *Questioni riguardanti la Geometria elementare*. Question 1020 runs as follows: "Consider the middle points C, C' of any two of the edges of a regular polyhedron. There is a plane regular polygon, having its centre at the centre of the circumsphere of the polyhedron, and its vertices at the middle points of certain of the edges, including C and C' . Then if this polygon have λ sides, and if m be the number of sides of each face of the polyhedron, n the number of edges of each polyhedral angle, c_λ, a_λ the side and in-radius of a λ -sided regular polygon, we have the following relations:

$$(c_\lambda/2)^2 = (c_m/2)^2 - a_n^2 = (c_m/2)^2 - a_m^2."$$

From this M. Hadamard obtains a new expression for the sides of the polyhedra inscribed in a given sphere:

$$2Rc_\lambda/c_n; \quad 2Rc_\lambda/c_m.$$

In some of the exercises the author introduces a *sorte particulière de géométrie*, in which some, but not all, of the relations between the elements of a triangle hold good. Other exercises give the elementary properties of quadrics, Dupin's cyclides, and the anallagmatic properties of the figure formed by two circles in space.

(12) We are glad to see a French translation of Mr. Russell's admirable essay on the Foundations of Geometry. The translation will, moreover, have a value of its own until we see a second edition of the English original, for it contains many corrections and additions, some on points of detail, and some in answer to the objections raised by critics. Unfortunately the reflections of the most eminent of these critics—M. Poincaré—reached the author too late to enable him to deal with them in this translation. Professor Halsted will be pleased to hear that Mr. Russell has abandoned his heresy *ré* the independence of Bolyai and Gauss. M. Couturat, whose excursions into lands no less distant than the region of mathematical infinity will be familiar to some of our readers, has enriched the book with a lexicon of philosophical terms, which will be welcomed by the mathematical reader; and Mr. Russell has added mathematical notes in an appendix, for which the average philosophical reader will probably not be quite so grateful. He deserves cordial congratulation on the interest aroused in both the mathematical and the philosophical worlds by his essay—an interest which is acknowledged in fitting terms in his prefatory note to the labours of Professor Cadenat.

(13) M. Laugel has translated into French D. Hilbert's famous *Grundlagen der Geometrie*. The author has endeavoured to lay down complete and simple groups of mutually independent axioms from which he deduces important geometrical theorems in such a way as to exhibit effectually the rôle played by these axioms, and the conclusions which may be logically drawn therefrom. The groups are five in number, and are classed as axioms of associa-

tion, distribution, congruence, and continuity, with the "axiom of parallels." The only spatial axioms are Nos. 3-7 in group I., all the rest are either linear or "planar." They show us more than ever the truth of the saying of M. Poincaré: all axioms are but definitions in disguise. But in the present case the disguise is easily pierced. After demonstrating by means of these weapons a particular case of Pascal's Theorem, and the ordinary laws for the theory of plane areas, we come to Desargue's theorem on the intersection of the joins of the homologous vertices of triangles whose homologous sides are parallel. We are shown that it is impossible to prove this theorem without the aid of either the axioms of congruence or of spatial axioms. From a new segmentary calculus based on this theorem we are led to the equation of a straight line, and ultimately to the construction of a Geometry of space. In the German edition the author does not deal with the possibility of discussing a Geometry without the axiom of parallels, or with points as elements coupled with the idea of groups of displacements as in Sophus Lie's "*travaux fondamentaux et féconds*." But a few interesting remarks on the subject have been added to this translation, and of these Prof. Halsted has made use in his review of Manning's Non-Euclidean Geometry, in the *Gazette*, No. 29, p. 94.

(14) The handsome volumes of M. Gournerie on Descriptive Geometry take us back to the days of Chasles and Poncelet. The first edition appeared in three parts from 1860 to 1864. Poncelet alludes to it in his *Traité des propriétés progressives des figures* as the most complete, the most accurate, and "*le plus rationnel*" of any work on this subject that had as yet appeared. Chasles became its sponsor in a very practical way, by speaking highly of it before the *Académie des Sciences*. The third edition of the third part has been edited by Professor Lebon, of the Lycée Charlemagne, who succeeded the author in his chair in the *Conservatoire des Arts et des Métiers*, and who is, moreover, the author of a large work on the same subject. Among the more interesting features of the part which has just appeared, we may mention the simple and elegant presentation of the theory of Curvature of Surfaces. In this, as well as in the proof of Euler's formula giving the curvature of a normal section, the infinitesimal calculus is not used, the author founding his treatment on Bertrand's Theorem (Salmon, *Geom. of Three Dim.*, p. 265, note). We find Meunier's Theorem (*loc. cit.*, p. 256) applied to the construction of the radii of curvature and the osculating planes of a curve given by its projections. The author follows Dupin's treatment of the lines of curvature of surfaces of the second order. M. Lebon brings the book up to date by notes historical and illustrative.

Differential and Integral Calculus. By E. W. NICHOLS. Pp. xii., 394. 7s. 6d. 1900. (Heath, Boston.)

This is essentially a book for beginners, by which we mean that the author has laboured to present his subject in the clearest and simplest manner, removing "all obscurities and mysteries," and smoothing the path of the student generally. Accordingly there is more explanatory matter than is generally to be found in a book for "the undergraduate courses of our best Universities, colleges, and technical schools." In some ways a book of this type is very useful to the tyro, especially if he be a private student. But there is always the danger that in removing every statement that may cause the student to exercise his wits, we are losing an opportunity of stimulating his attention and cultivating a valuable faculty. Mr. Nichols is to be commended for the skilful way in which the geometrical, mechanical, and electrical applications are worked in throughout the book. The historical notes are concise and to the point. Some twenty pages are given to differential equations. The volume is well bound and printed, and the author's boast that he gives the reader a "clear and open" page is amply justified.

Differential and Integral Calculus for Beginners. By E. EDGER. Pp. vi. 253. 2s. 6d. 1901. (Nelson.)

Yet another Calculus "adapted to the use of Students of Physics and Mechanics," "shorn of all extraneous difficulties," providing the "physical student with a valuable engine of research," the student finding "no difficulties which cannot be overcome by application and perseverance"! Assuming on

the part of the reader only an elementary knowledge of Algebra and Geometry, the author proceeds at once to initiate him into the mysteries of co-ordinate geometry and of circular and potential functions. The proofs of trigonometrical formulæ required are relegated to an appendix. The differentiation of simple and complex functions occupy pp. 27-53; then follow two chapters on maxima and minima and expansions of functions. Two chapters on integration lead up to applications to geometrical, mechanical, and physical problems. Double and triple integrations and easy differential equations bring the book to a close. The treatment is throughout clear and simple; in fact, many of the problems are worked out with almost unnecessary detail of development. For instance, the moment of inertia of an ellipsoid drags through $2\frac{1}{2}$ pages, and requires neither "application" nor "perseverance" to follow its course. But, no doubt, this is quite necessary with students whose mathematical equipment is limited in character. We notice a curious mistake on page 101. Here the author brings the mean ordinate to $y = a \sin bx$ into relation with the maximum of alternating electric currents. He seems to think that "the maximum current is a little greater than $1\frac{1}{2}$ times the average current." A reference to Everett (Deschanel, part iii., p. 176) will show that this cannot be true. Has the author confused square root of mean square into mathematical mean?

The Elements of Hydrostatics. By S. L. LONEY, M.A. Pp. x. 248. 4s. 6d. (Cambridge University Press.)

The peculiar characteristics of Mr. Loney's works are so familiar to most teachers that it is unnecessary to dwell on them here. Suffice it to say that the *Hydrostatics* is no exception to the rule. The chapter on Centres of Pressure strikes us as more complete than is usually the case in an elementary work. The valves in the Condenser (p. 180) might be more convincing. Why not have substituted for this picture a simple machine in daily use such as the bicycle pump? (cf. Greenhill's *Hydrostatics*, p. 374). The sections dealing with curves of buoyancy and tensions of vessels are as simple as is necessary for ordinary students. We think that some definite reasons should be given why the theory of Hydrostatics here laid down should be even remotely applicable to other than the "perfect liquid," and that, in general, more illustrative matter is advisable referring to the machines in daily use. It is really remarkable how differently a student works at theories which have obvious practical applications. We *must* be wrong if we neglect any method likely to convince the pupil of the practical value and relevance of his investigations.

Éléments de Mathématiques Supérieures à l'usage des Physiciens, Chimistes et Ingénieurs, et des élèves des Facultés des Sciences. By H. VOOR. Pp. vii., 619. 10 fr. 1901. (Nony, Paris.)

Nancy is one of the few Universities in Europe where the study of Physics and Chemistry is so well organised that students can avail themselves of the services of a Professor of Mathematics to unveil those mysteries of his science which are indispensable to their course of instruction. This book contains about fifty chapters, each about the length of a lecture, beginning with equations containing two unknowns, and ending with linear differential equations. From easy determinants, the binomial theorem, surds and indices, the author proceeds to the discussion of the doctrine of series, prefacing his remarks on series by generalities on limits. The usual rules on convergence, etc., are followed by a detailed account of method of approximation to the sum of a series. Logarithms and the exponential theorem complete the section devoted to Algebra, with the exception of the few notes on elimination and series at the end of the book. Analytical Geometry is introduced by general ideas on units, etc., the examples being derived from Geometry, Mechanics, and Physics. Then come co-ordinates and the representation of lines and surfaces by means of equations. Elementary Differential Calculus with its geometrical applications leads up to the Integral, which is carried as far as the theorems of Ostrogradsky, Stokes, and Green. The book closes with some two hundred exercises on all the sections. This rapid summary will give an idea of the ground covered by these lectures, and as they have been "professed" by the author for several years, they no doubt represent the minimum that is required by those who wish to have at their disposal in the shortest possible period a mathematical armoury for prac-

tical application. The explanatory matter is full without being verbose, and the style is simple and clear. This volume will not detract from the reputation of Professor Vogt.

PROBLEMS.

[Owing to pressure of space, these must be discontinued until the solutions of Nos. 1-449 are completed.]

SOLUTIONS.

The following problems, published in Vol. I. are still unsolved: 171, 275, 279, 283, 326, 336-8, 341, 349, 370, 373-4, 376-9, 381-2, 389, 395-7. Solutions to these, or to problems in Vol. II., should be sent as early as possible.

The question need not be re-written; the number should precede the solution. Figures should be very carefully drawn to a small scale on a separate sheet.

356. [I. 17. a.] Find three consecutive numbers each of the form $\square + \square$. (Ex. $2248 = 42^2 + 22^2$; $2249 = 43^2 + 20^2$; $2250 = 45^2 + 15^2$.) I have six solutions with numbers under 1000, and eight with numbers between 1000 and 2000. Can anyone indicate more solutions between these limits? W. ALLEN WHITWORTH.

Solution by R. F. DAVIS.

There are seven solutions with numbers under 1000; namely, 72-4, 232-4, 288-90, 520-2, 584-6, 800-2, 808-10. Also there are ten solutions with numbers between 1000 and 2000; namely, 1096-8, 1152-4, 1224-6, 1312-4, 1600-2, 1664-6, 1744-6, 1800-2, 1872-4, 1960-2. Of these three, viz. 288-290, 1224-6, and 1600-2, include numbers (289, 1225, 1600) which although expressible as sums of squares are themselves square numbers: these sets, therefore, Mr. Whitworth rejected.

Particular formulæ are

$$\left. \begin{aligned} \{a(a+1)\}^2 + \{a(a+1)\}^2 &= 2a^4 + 4a^3 + 2a^2 \\ \{a(a+2)\}^2 + \{a^2-1\}^2 &= 2a^4 + 4a^3 + 2a^2 + 1 \\ \{a(a+1)+1\}^2 + \{a(a+1)-1\}^2 &= 2a^4 + 4a^3 + 2a^2 + 2 \end{aligned} \right\}$$

which gives four of the above solutions, including one of the rejected; and

$$\left. \begin{aligned} (2a^3 - a^2 - a)^2 + (2a^3 + a^2 + a)^2 &= 8a^6 + 2a^4 + 4a^3 + 2a^2 \\ (2a^3 - a^2)^2 + (2a^3 + a^2 + 1)^2 &= 8a^6 + 2a^4 + 4a^3 + 2a^2 + 1 \\ (2a^3 - a^2 + a + 1)^2 + (2a^3 + a^2 - a + 1)^2 &= 8a^6 + 2a^4 + 4a^3 + 2a^2 + 2 \end{aligned} \right\}$$

which gives two more of the solutions.

Solution by C. E. YOUNGMAN.

The sum of two squares is either $4x$ (both even), $4x+1$ (even and odd), or $8x+2$ (both odd); so that three consecutive sums must have the forms $8n$, $8n+1$, $8n+2$. And one of these must be divisible by 3; $\therefore$ by 9, for a square is either $9y$ or $3y+1$;

$$\therefore n = 9m, 9m+1, \text{ or } 9m+2. \dots\dots\dots(\text{A})$$

Also $8n = (2a)^2 + (2b)^2$; $\therefore a$ and b are both odd or both even, and

$$4n = (a-b)^2 + (a+b)^2 = 4c^2 + 4d^2;$$

$\therefore n$ is either a square (when $a=b$) or the sum of two unequal squares;... (B)

$$\therefore n = 4p, 4p+1, \text{ or } 8p+2. \dots\dots\dots(\text{C})$$

(A) and (C) combined show that n must have one of the forms: $36r$, $36r+1$, $72r+2$, $72r+10$, $36r+9$, $72r+18$, $36r+20$, $36r+28$, $36r+29$.

Now for solutions below 1000, $n < 125$; proceed as follows:

Write down the possible values of n ; strike out all that do not satisfy (B). [To be the sum of two unequal squares a number must be a prime of the form $4x+1$, or a product of such primes, multiplied may be by powers of 2 and squares.]

n				$4n+1$				$8n+1$			
	36	72*	108*		145				289		
1	37	73	109	5	149	293	437*	9*	297*	585	
2*		74				297*					
9	45	81	117	37	181	325	469*	73	361*	649*	
10		82		21*		329*					
18*		90				361				721*	
20	56*	92*		81				161*			
28*	64	100			257	401			513*	801	
29	65	101		117	261	405		233	521	809	

The numbers marked * being thus struck out, find the values of $4n+1$ for those that remain; if $8n+2$ be the sum of two squares, $4n+1$ is either a square or the sum of two squares; strike out as before those values which do not satisfy this condition; and, finally, write down the values of $8n+1$ for the numbers left and strike out those which are not of the form $\square + \square$, and we are left with the middle numbers of seven solutions; and in the same way between 1000 and 2000 may be found ten more solutions, viz.: $8n+1=1097, 1153, 1225, 1313, 1601, 1665, 1745, 1801, 1873, 1961$.

369. [K. 5. d.] If triangles $ABC, A'B'C'$ be in perspective (centre P , axis DEF), and be also orthologic—i.e. such that perpendiculars from the vertices of either on corresponding sides of the other concur (in Q and Q')—then PQQ' is a straight line perpendicular to DEF .

COR. 1. If PQ be points in the plane of a triangle ABC such that perpendiculars from Q on AP, BP, CP , cut BC, CA, AB on a straight line, then so also do perpendiculars from P on AQ, BQ, CQ ; and both the lines are perpendicular to PQ .

COR. 2. If $ABCD$ be four points determining a rectangular hyperbola, the tangent at D may be had thus:—Draw DE perpendicular to BD cutting CA at E , and DF perpendicular to CD cutting AB at F ; then DT perpendicular to EF is the tangent.

C. E. YOUNGMAN.

Solution by C. E. M'VICKER.

Consider P the vertex of a triangular pyramid standing on ABC . Then DEF is the line common to the planes $ABC, A'B'C'$. If now the plane $A'B_1C_1$ be drawn through A' parallel to ABC , its line of intersection $A'D_1$ with $A'B'C'$ will be parallel to DEF . The axis of perspective of $A'B'C'$ and $A'B_1C_1$ is therefore parallel to DEF . It is then only necessary to prove the theorem for the orthologic triangles $A'B'C', A'B_1C_1$ having the vertex A' in common.

Draw B_1I, C_1J perpendicular to $A'D_1$ meeting QB, QC' in S and T , the latter meeting $A'B_1, A'C_1$ in U and V . UV is clearly perpendicular to $Q.A'$ and therefore parallel to BC, B_1C_1 .

$$B_1S/B_1U = \sin SUB_1/\sin USB_1,$$

$$C_1T/C_1V = \sin TVC_1/\sin VTC_1,$$

and

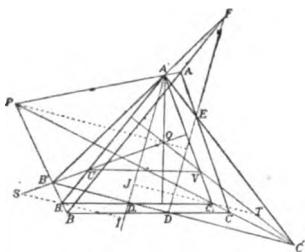
$$B_1U/B_1A' = C_1V/C_1A' \text{ by parallels;}$$

$$\therefore \frac{B_1S}{C_1T} = \frac{A'B_1 \sin SUB_1 \sin VTC_1}{A'C_1 \sin TVC_1 \sin USB_1} = \frac{A'B_1 \sin IA'B_1}{A'C_1 \sin IA'C_1};$$

$$[\because \sin SUB_1 = \cos A = \sin TVC_1]$$

$\therefore B_1S/C_1T = B_1I/C_1J$, so that S and T lie in directum with D_1 .

The sides of the triangles $PBC', \infty ST$ therefore pass through the collinear points $B_1D_1C_1$, their base angles lying on QB, QC' ; $\therefore$ the line ∞P joining their vertices passes through Q . Hence PQ is parallel to SI, TJ and therefore at right angles to DEF .



Otherwise: let $\alpha\beta\gamma$ and $xa, y\beta, z\gamma$ be vectors from the centre of perspective P to the vertices of $ABC, A'B'C'$, and let the vector $PQ \equiv \rho$. It is easily seen that

$$\sigma \equiv x(y-z)\alpha + y(z-x)\beta + z(x-y)\gamma$$

is a vector parallel to the axis of perspective of the triangles.

Since $\rho - \alpha$ is perpendicular to $y\beta - z\gamma$ we have, by quaternions,

$$S(\rho - \alpha)(y\beta - z\gamma) = 0.$$

Similarly,

$$S(\rho - \beta)(z\gamma - x\alpha) = 0,$$

and

$$S(\rho - \gamma)(x\alpha - y\beta) = 0.$$

Multiply in order by the scalars x, y, z and add, so as to eliminate terms independent of ρ ; then we obtain $S\rho\sigma = 0$, proving that ρ is at right angles to σ .

Mr. M'Vicker adds: I have not succeeded in deducing the corollaries from the proposition itself, but give in outline an analytical proof of a property suggested by them and including both.

$ABCDE$ are five coplanar points. Perpendiculars through E to DA, DB, DC meet BC, CA, AB in collinear points PQR ; it is required to prove that $ABCDE$ lie on a rectangular hyperbola, and that PQR is perpendicular to DE . Conversely, if they lie on a rectangular hyperbola, the perpendiculars from E to DA, DB, DC meet BC, CA, AB in collinear points.

Take as coordinates of the five points $(t_1, 1/t_1), (t_2, 1/t_2), (t_3, 1/t_3), (t_4, 1/t_4), (t_5, 1/t_5)$, the axes being the asymptotes of the rectangular hyperbola $xy = 1$ through the first four.

Form the equations of EP, EQ, ER ; P, Q, R being the intersections of BC, CA, AB with the line $ax + by = 1$: express that these lines are perpendicular to DA, DB, DC , and eliminate a and b ; we then find a determinant equation in θ which reduces to

$$\begin{vmatrix} t_1\theta + t_2t_3 & t_1t_5 + t_2t_3 & 1 \\ t_2\theta + t_3t_1 & t_2t_5 + t_3t_1 & 1 \\ t_3\theta + t_1t_2 & t_3t_5 + t_1t_2 & 1 \end{vmatrix} = 0.$$

This can only be satisfied by $\theta = t_5$ which proves that the fifth point lies on the hyperbola.

The converse is evident on putting $\theta = t_5$ at the beginning. If now we find $a:b$ from the equations which gave the determinant we get the ratio $t_4 t_5 : -1$, which shows that PQR is perpendicular to DE .

This latter property is also evident *a priori* since EP , EQ , ER are the sides of an evanescent triangle in orthologic perspective with ABC .

387. [K. 17. e. 20. f.] Construct a spherical quadrilateral given a, β, γ, δ , the mid points of the sides taken in order; and prove that the cosine of half its spherical excess is equal to

$$\cos \beta \gamma \cos a \delta - \cos \gamma a \cos \beta \delta + \cos a \beta \cos \gamma \delta. \quad \text{C. E. M'VICKER.}$$

Solution and Note by Proposer.

If $ABCD$ be the spherical quadrilateral, a, b, c, d the sides, DA, AB, BC, CD , and $E \equiv A+B+C+D-2\pi$ the spherical excess, then, by a known property of a triangle, $a\beta$ meets DB at a point ω distant $\frac{\pi}{2}$ from O the middle point of DB ; hence also $\gamma\delta$ meets DB at the same point ω . Also if ωX and ωP be taken along $\omega\beta, \omega\delta$ equal to $a\beta, \gamma\delta$ respectively; then the triangle ωPX is right-angled at P , and PX is half the spherical excess of ABD .

M'Clelland and Preston's *Spherical Trigonometry*, § 104.

Produce $a\beta, \gamma\delta$ to meet in ω ; and on $\omega\beta, \omega\delta$ take $\omega X, \omega Y$ equal respectively to $\beta a, \delta\gamma$; draw the great circle ωPBD perpendicular to XY ; mark off ωO a quadrant and BO, OD each equal to ωP ; then B, D are two vertices of the required quadrilateral, and XY is half its spherical excess.

$$\text{Again} \quad \cos \frac{1}{2}E = \cos XY = \cos \omega X \cdot \cos \omega Y + \sin \omega X \cdot \sin \omega Y \cdot \cos \Omega;$$

$$\therefore \cos \frac{1}{2}E = \cos a\beta \cdot \cos \gamma\delta + \sin a\beta \cdot \sin \gamma\delta \cdot \cos \Omega,$$

where Ω denotes the angle $X\omega Y$.

Take now the trigonometrical identity

$$\begin{aligned} & \sin(\lambda - \lambda') \sin(\mu - \mu') \cos \Omega \\ & \equiv (\cos \lambda \cos \mu + \sin \lambda \sin \mu \cos \Omega)(\cos \lambda' \cos \mu' + \sin \lambda' \sin \mu' \cos \Omega) \\ & \quad - (\cos \lambda \cos \mu' + \sin \lambda \sin \mu' \cos \Omega)(\cos \lambda' \cos \mu + \sin \lambda' \sin \mu \cos \Omega), \end{aligned}$$

and make $\lambda = \text{arc } \omega a, \lambda' = \omega\beta, \mu = \omega\delta, \mu' = \omega\gamma$;

then immediately we obtain

$$\sin a\beta \cdot \sin \gamma\delta \cdot \cos \Omega = \cos a\delta \cdot \cos \beta\gamma - \cos \gamma a \cdot \cos \beta\delta,$$

whence $\cos \frac{1}{2}E = \cos \beta\gamma \cdot \cos a\delta - \cos \gamma a \cdot \cos \beta\delta + \cos a\beta \cdot \cos \gamma\delta$, as was to be proved.

From this simple formula, E can be readily expressed in terms of the sides and diagonals of $ABCD$.

For by an elementary theorem,

$$\cos aA + \cos aB = 2 \cos \beta A \cdot \cos a\beta.$$

Multiply each by $2 \cos aA$ and denote DB, AC by e, f , then

$$1 + \cos a + \cos b + \cos e = 4 \cos \frac{a}{2} \cdot \cos \frac{b}{2} + \cos a\beta;$$

$$\therefore 2 \cos \frac{a}{2} \cos \frac{b}{2} \cdot \cos a\beta = \cos^2 \frac{a}{2} + \cos^2 \frac{b}{2} + \cos^2 \frac{e}{2} - 1.$$

$$\text{Similarly} \quad 2 \cos \frac{c}{2} \cos \frac{d}{2} \cdot \cos \gamma\delta = \cos^2 \frac{c}{2} + \cos^2 \frac{d}{2} + \cos^2 \frac{e}{2} - 1,$$

$$,, \quad 2 \cos \frac{b}{2} \cos \frac{c}{2} \cdot \cos \beta\gamma = \cos^2 \frac{b}{2} + \cos^2 \frac{c}{2} + \cos^2 \frac{f}{2} - 1,$$

$$,, \quad 2 \cos \frac{a}{2} \cos \frac{d}{2} \cdot \cos a\delta = \cos^2 \frac{a}{2} + \cos^2 \frac{d}{2} + \cos^2 \frac{f}{2} - 1,$$

similarly $2 \cos \frac{a}{2} \cos \frac{c}{2} \cdot \cos \gamma a = \cos^2 \frac{e}{2} + \cos^2 \frac{f}{2} + \cos^2 \frac{b}{2} + \cos^2 \frac{d}{2} - 2,$
 „ $2 \cos \frac{b}{2} \cos \frac{d}{2} \cdot \cos \beta \delta = \cos^2 \frac{e}{2} + \cos^2 \frac{f}{2} + \cos^2 \frac{a}{2} + \cos^2 \frac{c}{2} - 2.$

But by the preceding theorem,

$$\cos \frac{1}{2} E = \cos \beta \gamma \cdot \cos \alpha \delta - \cos \gamma a \cdot \cos \beta \delta + \cos \alpha \beta \cdot \cos \gamma \delta,$$

hence after reduction we find

$$4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2} \cos \frac{d}{2} \cdot \sin^2 \frac{1}{4} E \\ = \sin^2 \frac{e}{2} \sin^2 \frac{f}{2} - \left(\cos \frac{a}{2} \cos \frac{c}{2} - \cos \frac{b}{2} \cos \frac{d}{2} \right)^2,$$

the formula given by Serret, *Méthodes de Géométrie*; also Casey, § 119; M'Clelland and Preston, § 186.

As far as I am aware this latter result has not hitherto been obtained without the aid of stereographic projection, nor has the theorem in this article been given.

388. [K. 11. e.] *The line ABCD is cut harmonically in B and C; on AC and BD as diameters circles APC, BQD are described whose planes intersect at right angles, and PQ, PB, QC are drawn. Shew that*

$$PQ \cdot BC = PB \cdot QC.$$

A. S. TOLLER.

Solution by W. F. BEARD.

Bisect AC, BD at E, F. Draw a $\odot$ on AC as diameter in the plane of the paper.

From F draw FG $\perp$ r. to AD meeting PC produced in G.

With centre G, radius GB, describe a sphere cutting PC in H, K.

$$GC \cdot GP = \text{sq. of tangent from } G \text{ to } \odot APC$$

$$= GE^2 - EC^2$$

$$= FG^2 + EF^2 - EC^2 \dots\dots\dots \text{I. 47.}$$

$$= FG^2 + FB^2 \quad \text{for } EF^2 = EC^2 + FB^2 \text{ see (a)}$$

$$= GB^2 \dots\dots\dots \text{I. 47.}$$

$$= GH^2;$$

$$\therefore PH \cdot CK \text{ is a harmonic range.}$$

Also since $GB^2 = FB^2 + FG^2$; $\therefore \odot$ on BD as diameter at right angles to the plane of the paper is a section of the sphere, centre G, radius GB.

Thus Q lies on this sphere;

$$\therefore PHCK \text{ is a harmonic range;}$$

$$\therefore PQ : QC = PH : HC \quad \text{from } (\beta), \text{ (see below)}$$

and

$$PB : BC = PH : HC$$

„ „

$$\therefore PQ : QC = PB : BC$$

or

$$PQ \cdot BC = PB \cdot QC.$$

Q.E.D.

(a) If ABCD is a harmonic range, the $\odot$ s. on AC, BD as diameters cut orthogonally.

(β) If ABCD is a harmonic range, and P any point on the $\odot$ on AC as diameter, then $PB : PD = BC : CD = AB : AD$.

Slide Rule.—Professor Barrell informs us, with reference to his paper (Vol. II., No. 29) on the Slide Rule, that Messrs. A. G. Thornton, of Manchester, have already prepared a rule with the index line on the cursor 'broken,' as he suggested. The rule also gives an extra $\frac{1}{16}$ of 'dead end' at each end of the scale.

BOOKS, ETC., RECEIVED.

First Stage Building Construction. By B. CUNNINGHAM. Pp. viii. 240. Organised Science Series. 2s. 1901 (W. B. Clive).

**Plane Geometrical Drawing.* By R. C. FAWDRY. Pp. xi. 185. 6s. net. 1901 (Spon).

Algebra for Junior Students. By TELFORD VARLEY. Pp. vi. 152. 1s. 1901 (Allman).

**The Elements of Euclid.* Book XI. By R. LACHLAN. Pp. 51. 1s. 1901 (Arnold).

**A South African Arithmetic.* By H. HILL. Pp. x. 328. 1900. 3s. 6d. (Methuen).

Leçons d'Algèbre. By MME. A. SALOMON. Pp. 181. 2 fr. 1901 (Nony, Paris).

Problèmes de Baccalauréat. By H. VUIBERT. Pp. 524. 3rd edition. 1901. 5 fr. (Nony, Paris.)

Traité de Mécanique. By E. CARVALLO. Pp. 156. 1901. 2nd edition. 2 fr. 50. (Nony, Paris.)

Euclid XI. By R. LACHLAN. Pp. 50. 1s. 1901 (Arnold).

**Linear Groups with an exposition of the Galois Field Theory.* By L. E. DICKSON. Pp. x. 312. 12 m. Geb. (Teubner, Leipzig.)

* Will be reviewed shortly.

PROCEEDINGS OF THE

EDINBURGH MATHEMATICAL SOCIETY.

Vol. XIX., pp. 4, 77,

Session 1900-1901 (7/6). Williams and Norgate.

CONTENTS:—Allardice, R. E., *On four circles touching a common circle. On the nine-point conic. On a cubic curve connected with the triangle*; Carslaw, H. S., *Oblique Incidence of a Train of Plane Waves on a semi-infinite Plane*; Chrystal, G., *Elementary Theorems on Surds*; Davis, R. F., *Focal Relations of a Bicircular Quartic*; Gibson, G. A., *An extension of Abel's Theorem on the continuity of a Power Series*; Jack, J., *Change of Axes. Director Circle*; Mair, D., *On the n^{th} root of a prime*; Muirhead, R. F., *Inequalities*; Third, J. A., *Triangles triply in perspective*; Tweedie, C., *Area of Triangle in Cartesians. Addition Theorem in Trigonometry.*

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."--BACON.

President :

J. FLETCHER MOULTON, K.C., M.P., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.

R. B. HAYWARD, M.A., F.R.S.

Prof. W. H. H. HUDSON, M.A.

R. LEVETT, M.A.

Professor A. LODGE, M.A.

Professor G. M. MINCHIN, M.A., F.R.S.

Treasurer :

F. W. HILL, M.A., City of London School, Victoria Embankment,
London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.

H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

R. F. DAVIS, M.A.

J. M. DYER, M.A.

Professor R. W. GENESE, M.A.

G. HEPPEL, M.A.

F. S. MACAULAY, M.A., D.Sc.

C. E. M'VICKER, M.A.

J. C. PALMER.

Professor H. W. LLOYD TANNER,
M.A., D.Sc., F.R.S.

A. E. WESTERN, M.A.

C. E. WILLIAMS, M.A.

THE MATHEMATICAL ASSOCIATION, formerly known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all masters and mistresses, who are interested in promoting good methods of mathematical teaching. Such an Association should become a recognized authority in its own department, and should exert an important influence on methods of examination.

General Meetings of the Association are held in London once a term, and in other places if desired. At these Meetings papers on elementary mathematics are read, and any member is at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

"The *Mathematical Gazette*" is the organ of the Association. It contains—

(1) ARTICLES, on subjects within the scope of elementary mathematics.

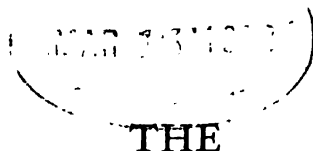
(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books.

(3) REVIEWS, at present the most striking feature of the *Gazette*, and written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell rather on the general development of the subject, than upon the part played therein by the book under notice.

(4) PROBLEMS AND SOLUTIONS, generally selected to show the trend of investigation at the universities, so far as is shown in the most recent scholarship papers. Questions of special interest or novelty also find a place in this section.

(5) SHORT NOTICES, of books not specially dealt with in the REVIEWS.

Members are requested to send any change of address to one of the Secretaries, with whom also intending members are requested to communicate. The subscription to the Association is 7s. 6d. per annum, and is due on Jan. 1st. It includes the subscription to "The *Mathematical Gazette*."



Sa 870.94

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 31.

JANUARY, 1902.

One Shilling Net.

The Mathematical Association.

THE GENERAL MEETINGS will be held at
King's College, Strand, London, W.C., at *Eight*
p.m., on *Thursday, May 8*, and *Thursday, October 2*, 1902.

The attention of Members and Subscribers is drawn to the Special
Notice on the back cover.

The Mathematical Gazette is issued in January, March, May, July,
October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, Alford House, Aberdeen.

All other correspondence should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

Secretaries of Association.

NOTICE.—The Index and Title-page to Volume I. can now be obtained from the Publishers.

CONTENTS.

THE ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION.

DISCUSSION ON REFORM IN THE TEACHING OF MATHEMATICS.

	PAGE		PAGE
Professor A. Lodge, - - -	129	Dr. F. S. Macaulay, - - -	139
Professor E. M. Minchin, - - -	132	Mr. C. Godfrey, - - -	140
Professor W. H. H. Hudson, - - -	134	Mr. T. Wilson, - - -	141
Professor M. J. M. Hill, - - -	135	Mr. T. J. Garstang, - - -	141
Mr. F. E. Marshall, - - -	137	Professor A. Lodge, - - -	142
Mr. E. M. Langley, - - -	138		
COMMUNICATION FROM TWENTY-THREE MASTERS IN PUBLIC SCHOOLS, -	143		
REVIEWS AND NOTICES, - - - - -	146		
BOOKS, ETC., RECEIVED, - - - - -	147		

Will appear shortly :

The Trigonometry of the Tetrahedron. (Rev. G. Richardson, M.A.)

Tripolar Coordinates. (G. N. Bates, M.A.)

Notes on Conics in Areal. (T. J. I'a Bromwich.)

Reviews by Professor E. B. Elliott, F.R.S.; Professor G. B. Mathews, F.R.S.; Professor H. W. Lloyd Tanner, M.A.; T. J. I'a Bromwich, M.A.; W. J. Dobbs, M.A.; J. H. Grace, M.A.; E. M. Langley, M.A.; W. H. Macaulay, M.A., and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. No. 8 is out of print. A few numbers may be obtained from the Editor at the following prices, post free :

No. 7, 4s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 14s.

These will soon be out of print and very scarce.

MAR 15 1902

THE
MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

JANUARY, 1902.

No. 31.

ANNUAL MEETING, 1902.

DISCUSSION ON REFORM IN THE TEACHING OF MATHEMATICS.

THE Annual Meeting of the Mathematical Association was held at King's College, London, on Saturday, January 18th. In the absence of the President, Mr. J. F. Moulton, K.C., the chair was taken by Professor E. M. Minchin, Vice-President. Professor Henrici wrote regretting that he had been prevented from attending. There were 29 members present. The subject discussed, "Reform in the Teaching of Mathematics," was introduced by Professor A. Lodge.

The Chairman, in opening the meeting, said they were prepared for as good an exhibition of Donnybrook Fair as could be given, for there were sure to be many teachers of mathematics present who would not agree with some of the points discussed by Professor Lodge. The fundamental question in the reform of mathematical teaching was the reform of geometry.

Professor Lodge then read his introductory paper.

MR. PRESIDENT, LADIES, AND GENTLEMEN,

The subject of my paper needs no apology before such an audience as this, for the Mathematical Association exists for the purpose of inaugurating and furthering improvements in mathematical teaching. The special object in bringing the whole question forward now is to enable us to co-operate with the British Association Committee formed for the same purpose at the Glasgow meeting last year.

Many teachers have been for a long time aware that the teaching of geometry in this country was suffering from its being based on a fixed, ancient model, which, however excellent, was not in many respects satisfactory as a text-book for beginners. Hence the formation of the A.I.G.T., of which Association we are the lineal descendants. The efforts of the Association were, however, powerless to make any appreciable effect on the action of the great examining bodies in the country, and without their co-operation much progress was not possible.

Now, however, with the powerful leverage of the British Association to assist us, we may confidently look for real and lasting progress.

The best method of teaching geometry will no doubt be the question which will require most attention, as that is a matter in which both teachers and examiners must move together if at all. But there are points in which

improvement might be asked for in connection with arithmetic and algebra. I wish chiefly to consider the geometrical question, so I will touch on only a few points. For example, men come up to engineering colleges who are slow and inaccurate in computation, who do not know the contracted methods of multiplication and division, who are as likely as not to put the decimal point in the wrong place. We want boys taught to be ready and rapid computers, to be able to make rough checks on their own work so as to avoid gross errors, to cultivate common sense in connection with problems, and to be in the habit of verifying answers.

This applies to algebra as well as to arithmetic, even more so to algebra perhaps. A boy will perform an absurdity in algebra, quite unconscious of its absurdity, which he would never think of perpetrating in arithmetic.

Thus we often find a student writing $\frac{1}{a} + \frac{1}{b} = \frac{1}{a+b}$ who sees at once the absurdity of $\frac{1}{2} + \frac{1}{3} = \frac{1}{5}$. His algebraic work should grow out of arithmetic, and continually be brought back into touch with it by numerical checks.

He should also be taught to multiply and divide rapidly by the method of detached coefficients, and be drilled in short division (synthetic division) in the case of simple divisors.

Then, again, arithmetical work can be so directed as to prepare the boy for geometry, by actual measurement and calculation in connection with geometrical figures, both plane and solid. The ratio of the circumference of a circle to its diameter could be verified by measurements on a cylinder. The area of a rectangle, parallelogram, triangle, etc., should be numerically worked in connection with scale drawings or actual solids, and verified by squared paper. Volumes of actual solids should be calculated from the pupil's own measurements. The eye might be trained to guess areas and volumes and the guesses be verified by actual measurement and calculation.

The algebraic formula $(a+b)(c+d) = ac+ad+bc+bd$ could be illustrated by rectangles drawn on squared paper. A very pretty illustration of this formula is afforded by the duodecimal multiplication of feet and inches by feet and inches, the algebraic answer coming out in square feet, inch-feet, and square inches.

The pupil should learn at an early stage to measure angles in degrees, and to learn by experiment such things as that the angles of a triangle add up to two right angles. The angles of various triangles could be estimated by eye and then measured. The notions of complementary and supplementary angles could be easily introduced, and the angles connected with a transversal crossing parallel lines be considered. Herbert Spencer's *Inventional Geometry* in the hands of a good teacher would be invaluable in stimulating the energies and intellect of a junior class.

When the pupils have thoroughly grasped elementary geometrical notions, and been practised in measurements and calculations, then they would be ready for a course of deductive geometry.

And here it must be remembered that the pupil's mental equipment is chiefly arithmetic and algebra, and his geometry should be built on these notions as much as possible, instead of being carefully divorced from them as is done in so many text-books. In Euclid's time arithmetical work was abstruse and difficult; in fact, Euclid approached arithmetic through geometry. We have to do the reverse, and we ought to do it wholeheartedly and without reserve.

I believe we could not do better at the outset than adopt some French text-book as our model. The Americans have done so already.

The chief points in the French text-books are :

- (1) The more orderly arrangement of propositions.
- (2) The entire separation of theorems from problems of construction, hypothetical constructions being used in proving a theorem.

(3) The closer association of a proposition and its converse when both are true.

(4) The adoption of arithmetical notions and algebraic processes.

(5) The early introduction of simple loci.

(6) Insistence on accurate figures drawn by accurate and practical processes.

(7) Practice in exercises from the very beginning.

Mr. Greenstreet suggests that I should also add :

(8) Attention paid to the various phases of a theorem as the figure changes, and (as the student progresses) to the easier forms of generalization.

The greater part of these improvements could be adopted at once, provided the sanction of the great examining bodies can be obtained. The first is the one which presents the greatest difficulties, but many of these difficulties would disappear with the sanction of No. 2.

Of the importance of a rearrangement there can be no doubt.

For example, the propositions that

Two triangles are equal if two sides and the included angle in one are equal to two sides and the included angle in the other, and that two triangles are equal if two angles and the included side in one are equal to two angles and the included side in the other,

are obviously related propositions.

In the French text-books they are juxtaposed, both being proved by superposition. Also I. 24, 25, being obviously related to I. 4, are made to immediately follow it in such of the French books as define a straight line to be the shortest distance between two points.

Again, the theorems relating to angles, such as I. 13, 14, 15 should come before triangles, as they do in the French books.

The whole subject of rearrangement is too vast to be treated in the course of a paper—it must be settled by a committee, or at least its most important features would have to be so settled ; and the sanction of the great Examining Boards must be obtained.

With regard to the fourth point above, there is nothing in the French geometries corresponding to the greater part of Book II. ; being rendered unnecessary by the arithmetical and algebraic treatment of areas. They could be taken as exercises in areas. The only theorems to be retained are II. 12, 13, which can thus immediately follow I. 47, 48, with which they are obviously connected.

[Euc. II. 11, 14 are merely particular cases of the graphic solution of quadratic equations by the method given in Cremona's *Geometry*, which should I think be more generally known and taught. However, these are problems, and therefore not in the category to which we are now attending. So also with the whole of Book IV.]

The French treatment of areas forms a simple introduction to Book VI. They prove that the measure of the area of a parallelogram is the product of its base and height, and that the area of a triangle equals half this product. Hence triangles (or parallelograms) on equal bases have areas proportional to their heights ; and those of the same height have areas proportional to their bases.

There is one other suggestion which I should like to make, and that is that the student should be encouraged in anticipating theorems, so as to have his mind prepared for a new theorem before being set to formally prove it ; and he should test the truth of new theorems experimentally, using accurately drawn figures.

In conclusion, I would urge on all those who are convinced that reform in geometrical teaching on some such lines as I have indicated is urgent and imperative, that they should not rest content until some at least of the

reforms are sanctioned by the great public examining bodies, and I think this meeting should not conclude without appointing a strong committee to co-operate with the British Association Committee and assist it in every way possible. That Committee has already had a valuable communication* signed by 23 schoolmasters, and has asked me to express to this Association its request for the fullest co-operation and advice.

The Chairman said the main point in the question of the teaching of geometry was the problem of dealing with the great waste of time that takes place with the subject. For some reason or another the boy who begins to learn geometry at nine makes hardly any progress by the time he is 14. He goes probably to a preparatory school at nine, and stays there till about 14, when he goes to one of the public schools. During that time he may succeed in getting through two books of Euclid, and some clever boys get over three. That is a very miserable amount of work for five years. What is the cause of it? He had come to the conclusion that the cause was the adoption of Euclid's language and method. The schoolboy is not taught geometry; he is taught to remember the words of Euclid. Mr. Lodge emphasised the teaching of arithmetic and algebra and their use in the teaching of geometry. There are people who say you must not teach arithmetic or algebra in connection with geometry. A few weeks ago he examined a book in which the principle was laid down that those who are doing Euclid may be allowed to save space when speaking of the square on the line AB by writing AB^2 , but it was merely as a notation for "the square on the line AB ," and it was stated that as soon as the boy conceived that it meant any arithmetical quantity, its use was to be immediately stopped. The boy might write "rectangle AB, CD ," but the moment he got an arithmetical notion from that, the notion was to be driven out of him. That idea is probably responsible for the whole waste of time that takes place in the teaching of geometry.

Professor Lodge recently showed him (the speaker) a little Belgian book for the secondary teaching of young girls in Belgium. It was a course of one year's teaching. There is more than six books of Euclid in it, and all through the algebraical and arithmetical notation is used; hence the rapid progress.

There must be several present that afternoon who had had experience of the public examinations: those who have not would, he thought, not believe him if he told them of some of the gross absurdities that examiners in Euclid have to deal with. He would just mention two cases to show them the kind of intellectual process that the learning of Euclid means for the ordinary schoolboy.

At one of the public examinations one boy who came from a very great public school was answering the question, "What are the cases enumerated in Euclid in which two triangles are identical?" The answer expected was: "The 4th, 8th, and 26th propositions of the 1st book"; but this ingenious young gentleman gave 13 cases in the 1st book of Euclid, and he made some of them up in this way: he said, "If there are two triangles, ABC, DEF , and the sides AB, AC of one are equal to DE, DF of the other and the included angles are equal, then the triangles are equal. Also, if AB, BC of the one are equal to DE, EF of the other and the included angles are equal, then the triangles are equal," and so on for 13 cases. Now, such an answer does not indicate an intellectual process.

At the same examination a boy was required to draw a perpendicular to a given line at a given point. He said, "Let AB be the given line and C the given point. Draw CD $\frac{17}{8}$ Then shall CD be perpendicular to AB ; for, if not, let CE be drawn perpendicular to AB ," etc. Now, that again is not an intellectual process: it is nothing but the repetition of a mere jargon.

* V. p. 143.

He could give other instances of erratic thought as well as in geometry, but no other branch of mathematics was so full of humorous episodes. One story he might relate seemed to convey a reflection on his own ability to teach statics. He asked a student to define a funicular polygon of a number of forces. The definition he obtained was, "A funicular polygon is a kind of polygon very difficult to draw, and impossible to understand." (Laughter.)

What Mr. Lodge had said about graphic solutions was worthy of attention. These should, in his opinion, accompany every branch of mathematics, and not be restricted to geometry: he hoped that graphic methods of solution would be generally adopted.

That concluded his remarks about the teaching of geometry. There was another question, the reform of teaching of other branches of mathematics—*e.g.*, dynamics and hydrostatics. Some of those present had to teach dynamics as well as geometry. There had been two methods of teaching dynamics in vogue. One was to begin with the teaching of statics and then to pass on to the teaching of kinetics, or what used in the old days to be called dynamics. Thomson and Tait many years ago advocated a different plan. They made no distinction. Force was one and the same thing in both subjects. From the former method the student got an idea that there were two different kinds of force: it was difficult to get this idea out later. Some people tried to make no distinction; they taught the two subjects together. He had recently done it himself, and he found that, taking the two subjects together *ab initio*, more rapid progress was made and sounder ideas inculcated in dynamics than under the old method. With regard to the way in which this subject is to be treated, he thought that those teaching it ought to remember that in the whole subject of statics there are just *two* fundamental facts, which are: First, that if you have any number of vectors, the resultant of these vectors has the same component along any line as the vectors themselves; and, secondly, that the moment of the resultant vector about any axis is equal to the sum of the moments of the vectors about that axis. That ought to be kept prominently before the student; and its importance becomes more pronounced in the most advanced parts of kinetics.

With regard to hydrostatics, he supposed the subject is taught in all the schools; yet he found with the students who came up to the Engineering College at Cooper's Hill that it was impossible to get any accurate hydrostatical notions from them. He sometimes asked them what was the number of cubic inches contained in so many lbs. of a given metal, the specific gravity of which was given in the table of specific gravities provided. Now the students knew that $W = Vs$: they had learnt this formula; and when asked, "How many cubic inches are there in 10 lb. of platinum?" they put $10 = 22V$, 22 being given as the specific gravity of platinum in the table. He (the Professor) asked them: "Is this $\frac{10}{22}$ cubic yards, or miles, or what?" They often reply "Feet," and on being asked the reason say, "Because a cubic foot goes so well with a pound." (Laughter.) There is very great difficulty in getting accurate results in any subjects from students whose attention has not been riveted on the fundamental principles of that subject.

He passed on to kinetics. He supposed they would all say that one of the most difficult subjects was the subject of rigid dynamics. In that subject, if one took up a text-book and opened it at almost any point, one would probably see large clouds of symbols which are quite unnecessary. In the deduction of the great dynamical principles, of which there are three, the principles should be deduced directly from Newton's second and third axioms, and one does not require to write down a single symbol. It follows from the second axiom, the internal forces being equal and opposite in the same right line,

that the resultant acceleration of each particle coincides with the resultant of all the forces internal and external acting on that particle. Now, making use of the two fundamental properties of vectors above mentioned, we have at once the principle of the motion of the centre of mass, and also the principle of the moment of the momentum of any system about any axis. You have then at once an explanation of what one supposes to be one of the most wonderful things ever seen, the motion of the gyroscope.

There is another thing: we have not yet succeeded in eliminating the principle of D'Alembert from our books. We ought ruthlessly to cut it out of our treatises on dynamics. This principle of D'Alembert is contained in the 2nd and 3rd axioms of Newton. The expression that D'Alembert gave it was one in which he introduced the notion of fictitious forces, and one result of this is the fallacious conception of centrifugal force. This he (the speaker) considered to be one of the great physical fallacies. Most people think centrifugal force to be a force tending to drive a body revolving round a centre away from that centre. There is no such force; and it is this principle of D'Alembert which is responsible for the fallacy.

Finally, this Association might possibly do something towards reform of the nomenclature of dynamics. The British Association appointed a committee for dealing with the nomenclature of physics. A similar committee would seem to be required for revising the nomenclature of applied mathematics. Thus, we speak about the *angular momentum* of a system about an axis. Angular momentum is no more momentum than chalk is cheese: it is in reality *moment of momentum*. Another expression is the term *moment of inertia*. What is the moment of inertia of a mere area, which has no mass and no inertia? A conception to which he found students take kindly is the conception of the "mean square of distance from any axis." The radius of gyration is simply the square root of the mean square of distance of a body, area, or volume from an axis.

The necessity for a reform of nomenclature is great in dynamics and in physics too. The physicists seem to be quite happy with such terms as "electromotive force" and "magnetomotive force." Now electromotive force and magnetomotive force ought to mean some kinds of Force. Each term means nothing of the kind. This association should take up the question of reforming other branches of mathematics. It is in the interests of young boys that it should deal with the reform of geometrical teaching: we want to be able to produce some valuable result at the end of five years' teaching instead of the wretched results attained by teaching Euclid's very words and order.

Finally, he would like to ask the question, "What is a Circle?" In an American book and in an English book he had seen it described as the *area* inside what he had been in the habit of calling a circle. The French distinguish between a circle and the circumference. Euclid did not do so. Was the circle then a curve or an area? The difficulty was in the preposition—*on* a circle, or *in* a circle. Euclid, though he defines it as an *area*, uses it as a curve.¹

Professor Hudson confined himself to the teaching of geometry. He conceived there were two main things to reform in the teaching of geometry. First, to induce teachers to prepare for the teaching of reasoned geometry by a preparation such as that given in Spencer's book. The pupil should become aware of the *facts* of geometry before reasoning about those facts. He was not so much inclined to complain of the slowness in the initial stage to which their chairman had referred. It does require time to lay and settle a good foundation. Slowness in the early stages is compensated for by rapidity afterwards. On this teachers should hold their

[¹ For this point in American and British Geometries v. *Gazette*, p. 118, l. 21 up, and p. 119, l. 9 up. W. J. G.]

own firmly against the parents. Time is *not* wasted though one may not get to the end of the first book in the first year.

The second main point for reform is that of the acceptance of propositions learnt by heart. That goes on to a terrible extent. Pupils can succeed in passing examinations by means of it. Now, it should not be possible to pass an examination in geometry by the mere reproduction of propositions learnt by heart. No alteration in the text-book will bring about a reform if that is to be allowed to continue. If the French or American books were used, and the pupils learnt by heart, and were allowed to pass examinations thereby, we should have all the same trouble as now over again. He agreed with nearly all that Professor Lodge said, but perhaps not with all he *meant* with regard to the use of algebra in geometry. He was strongly of opinion that the second mathematical subject to be taken up was geometry. Arithmetic, geometry, algebra is the correct order. The child does something of arithmetic at the age of two or three, and some progress, slight though it may be, is made in the subject. Geometry then may be taken at from 7 to 9, but algebra could scarcely be begun till a couple of years later. It requires a more mature mind than the other subjects. He would be sorry if a reform in geometry were to drive out the subject as a distinct branch by itself. The two subjects can be carried on side by side, and made to illustrate one another, if the pupils know algebra; but that algebraical proof should be substituted would be very unfortunate and unnatural in the early stages.

Some of his (the speaker's) pupils, who were young ladies, having no notion of algebra, had derived great benefit from pursuing the study of geometry, and doing it well. He would be sorry if any change could render that impossible.

There were various minor points to which he would wish to refer. Euclid's definition of an angle should be altered, and the modern definition adopted in its place. In the third book he thought the doctrine of limits must not be evaded. The modern definition of a tangent is implied in one of Euclid's propositions. Another notion Professor Lodge had referred to was the notion of a "locus." That notion should be introduced early, as should the notion of "symmetry." In the second book, the new notion of "sense" along a line might be treated so as to be understood by a person ignorant of algebra, and, therefore, ignorant of the idea of a "negative" quantity. He wished this treatment of the thing in geometry to be preparatory to the idea of "minus." A great general principle is that teaching should all be anticipatory, and clear the way for some after teaching. That is illustrated by the doctrine of "sense" along a line preparing for the conception of a negative quantity, the definition of a tangent preparing for subsequent fuller treatment of the doctrine of limits, etc. Let our early teaching prepare the way for the subjects which have to come after. They must try to induce teachers to begin with the actual handling of solids. Plane geometry cannot be begun before the age of seven. It is dangerous to put compasses in the hands of very young people.

Professor Hill said that all teachers of mathematics should be grateful to Professor Perry for bringing the subject of the teaching of elementary mathematics before the British Association, and to Professor Lodge for asking the Mathematical Association to discuss it. The concrete representation of problems, wherever possible, which Professor Perry ably advocated, was of the greatest service. Most teachers of mathematics cannot help admitting that there must be something wrong in the teaching of that subject when the results of that teaching are so poor. In seeking for the cause of the failure, the speaker had come to the conclusion that it was to be found in the fact that too great a strain was placed upon the memory. From the very beginning of the subject too many rules were taught and too

little attention was paid to principles. This was a matter upon which the speaker was at direct issue with Professor Perry, and, indeed, there were some passages in Professor Perry's treatise on Practical Mathematics which seemed to show that Professor Perry was at issue with himself. For example while on the one hand on page 120, he says, "Rules in mensuration ought to be stated as formulae, and proved if the proofs are easy, as part of the geometrical work," he says on the other hand (v. p. 22), "Have you not noticed that a great man has only a few simple principles on which to regulate all his actions? A great engineer keeps in his head just a few simple methods of calculation." These two passages from Professor Perry's book state very clearly two different modes of teaching, the one resting on *rules* learned by heart, the other on *principles* and *methods*.

Unfortunately, Professor Perry seems to incline to the method of teaching by rules. He says in his address to the educational science section of the British Association, "Why should not a boy assume the truth of many propositions of the first four books of Euclid, letting him accept their truth, partly by faith, partly by trial? Giving him the whole fifth book of Euclid by simple algebra. Letting him assume the sixth book to be axiomatic."

It is this method of teaching by rules that is responsible for the failure of the majority of young people to determine with accuracy the position of the decimal point in a quotient.

Professor Lodge seemed to the speaker to lay too great stress upon the acquisition of rapidity in calculation. Useful as the faculty is, it is not sound policy to sacrifice a thorough comprehension of principles to its acquisition. The main thing in the present circumstances, however, was to develop a constructive policy, and this was perhaps more necessary in the case of the teaching of geometry than in other subjects. Euclid's order is a cause of great difficulty. He seems to have first obtained his propositions, and then after much reflection to have arranged them in groups without reference to the order of discovery. The speaker was once present at a prize distribution and heard the chairman, after first disclaiming all recollection of his mathematics, go on to say that there was one proposition which had left a great impression on his mind of the magnitude of Euclid's genius. This was the construction for an isosceles triangle, in which each of the base angles is double the vertical angle (Euc. IV. 10). On thinking the matter over, it seemed to the speaker that the probable course of ideas was this: Euclid started with the idea of constructing a regular pentagon. It at once became evident that the construction depended on the construction of an isosceles triangle of the kind mentioned above. To construct this, Euclid examined the relation between the segments of the side of any triangle made by the bisector of the angle at the opposite vertex (Euc. VI. 3). Having obtained this, he reduced the problem of the construction of the isosceles triangle required to the cutting of a straight line into two parts, such that the rectangle contained by the whole line and one part is equal to the square on the other part, and this he put into the second book, where it stands as the eleventh proposition.

This is perhaps one of the most striking examples of the artificiality of Euclid's order, though there are many others, some of which have already been mentioned by previous speakers. The real question is, What should now be done? The speaker believed that it was absolutely necessary to re-arrange the treatment of the subject in a more natural order, and that this order should be discovered by experiment. He had himself attempted to teach the subject to a child, and he was surprised at the amount of work that could be done with a pencil, a ruler, a pair of scissors, and a piece of paper. For example, he had cut out the angles of a triangle and put them together, and asked the child, "How much have you?"

The child replied "Half a 'round.'" He then took a quadrilateral and cut out the angles and put them together. The child called the result a whole "round." And so on with a pentagon the angles formed a round and a half. The next day the speaker took a triangle and produced the sides, and cut out the external angles. Putting these together the child said, "You have now a whole 'round,' the same as you had yesterday for a quadrilateral." The next step taken was to put together the external angles of a quadrilateral. Whilst this was being done, the child said, "You will get more now than for the triangle," thus showing that he had formed the idea that the sum of the angles must increase with the number of sides of the figure. When it turned out to be only a "round," the child asked that a five-sided figure should next be taken, and when the same result was reached, he asked that a ten-sided figure might be tried. When this again gave only a "round," the child seemed to see that the result would be always the same; and in this way the truth of the proposition was stamped upon his mind before he was able to understand, or even read, one of Euclid's demonstrations. Experimenting in this way, the speaker believed that it would be quite possible to discover a more natural order than Euclid's, which would present far less difficulty to beginners.

Before passing away from the subject of Euclid, the speaker wished to say something about the fifth book of Euclid. Professor Perry said, in his address to the British Association, that the whole of the fifth book of Euclid should be given by simple algebra. And there had recently appeared a letter in *Nature*, signed by 23 mathematical teachers, stating that no one now teaches the fifth book of Euclid. The speaker was willing to admit that this book, which, with the *Logic* of Aristotle, Professor De Morgan described as the two most unexceptionable and unassailable treatises ever written, was unsuitable for elementary teaching. For this there was a definite reason, viz., that Euclid uses his test for distinguishing between *unequal* ratios to prove properties of *equal* ratios. Now it is evident that if Euclid's test (Euc. V. Def. 7) for *equal* ratios (Euc. V. Def. 5) is a good and sound one, it should be possible to deduce from it all properties of equal ratios, without using the test for unequal ratios. The neglect to do this makes Euclid's proofs artificial and therefore difficult. When the whole argument is made to depend on his test for *equal* ratios alone, the whole of the difficulty of the book is swept away. The great merit of the book is that the treatment is applicable both to commensurable and incommensurable magnitudes. The algebraic treatment, of which Professor Perry speaks, is applicable only to commensurable magnitudes, and this, in the opinion of the speaker, should be given to boys as soon as they reach Euc. I. 35, 37 [parallelograms (triangles) on the same base, and between the same parallels, are equal to one another], and they then get for commensurable bases Euc. VI. 1 [triangles and parallelograms of the same altitude are to one another as their bases]. In like manner Euc. VI. 33 is reached through Euc. III. 26.

But this mode of developing the subject would not render it unnecessary to provide a place for the treatment of incommensurables at a not very advanced stage of instruction, and to that treatment there was no better introduction than Euclid's fifth book, after it had been modified in the manner indicated above. Incommensurables exist and must be dealt with, and in the opinion of the speaker they should be dealt with before the calculus is commenced, if exact ideas of the theory of limits are ever to be attained.

Mr. F. E. Marshall said it was some 14 or 15 years since he was last present. For more than 30 years he had been gaining experience as to what mathematics boys ought to have done between the ages of nine and fourteen. There were several things in connection with their subject to which he would like to refer. First of all, there was the exceeding badness of the Euclid teaching given to boys at that stage. They ought to recognize

the fact that boys are then taught by people who in many cases have had no mathematical training, whether they be assistant masters in the preparatory schools or governesses at home. He could not help thinking that to circulate a notice, describing the way in which they wanted the subject to be approached, might be productive of good. The thing that most wants saying to those elementary teachers of geometry is, "For goodness sake don't let your students see a text-book." He believed that much of the failure in geometrical teaching of which they complained to-day was due to the use of text-books and the want of some preparation in geometrical drawing. He thought, with Prof. Hudson, that there was a danger of being too numerical in the latter. Teaching without a book, the teacher could watch the pupil's mind as it grasped each new point put before him, with full profit to the pupil and a pleasure to the teacher, in great contrast with the profitless tedium of testing in writing what the pupil had learnt—and perhaps not half understood—from a text-book. In *visd voce* the pupil approaches a new theorem by the method of analysis—the natural method of discovery—and afterwards throws his proof into Euclid's synthetical form. He pleaded that some attention might be given to the teaching of arithmetic. He felt much might be got from attention to two or three simple things. They should try to get the elementary teachers to deal thoroughly with the decimal notation. They should go straight from decimal integers to decimal fractions, and not, as they so frequently do, introduce decimal fractions as a distinct subject *after* vulgar fractions. They ought to prepare the way for later teaching and the use of approximate methods. Thus, in multiplication, they should always multiply first by the highest digit. The pupils should be trained to see the working of the metric system, not merely the conversion from that system to another. He considered the teaching of rapidity of computation to be one of the most valuable things they did. By making a boy do the simplest processes with extreme rapidity, his power of fixing his attention is very quickly improved. His mind gives a quicker response to any external stimulus. With regard to algebra, as Professor Hudson says, it should be *third*, and the foundations for it should be soundly laid in the teaching of arithmetic. They, in public schools, received boys who professed to solve quadratic equations, and yet were unable to work out correctly simple questions in vulgar or decimal fractions. It seemed to him that they should try to formulate their desires in the matter of geometry, and then leave the matter of a text-book to come later. It took such a long time to get a text-book or syllabus agreed upon. Could they not get the examining bodies to sanction a little change, and so proceed little by little? If they could first get the examining bodies to cut out a few of the things they all objected to, they could then ask for more. He felt there was a great danger in asking too much at once.

Mr. E. M. Langley agreed cordially with the previous speakers so far as they agreed with one another. He especially agreed that they ought not to wait for big things, though of course he hoped they might influence examiners. They ought to set to work on experimental courses that would help to make boys see what their mathematics meant. They had, as the A.I.G.T., made a great mistake in asking that no text-book might be authorized—it was too revolutionary a step for the authorities. One of the chief difficulties would come with the teachers, especially with the junior teachers. It would be no good saying to them, "Go and teach experimental geometry straight off." They would say at once, "Very well, how are we to do it?" As to text-books, of course, the boys would be able to do without the text-book if the teacher could. (Laughter.) There could be no doubt that every branch of mathematics should be illustrated as far as possible from other branches. There are no water-tight compartments in mathematics. He used to be stared at when he first asked a boy solving equations, "What

axiom are you using there?" As if axioms were only used in Euclid. Boys should certainly be encouraged to test their answers. He did not know whether they knew Professor Perry's story of the postage stamp. A Cambridge undergraduate was asked how many postage stamps would be required to cover the walls of a room—the Senate house, perhaps—and after a great deal of work the answer 1, with a decimal running to some twenty or more places, was produced. (Laughter.) He had one of his own almost as good. When he (the speaker) started teaching the metric system he held up a boxwood metre and, explaining to the class that a kilometre was a thousand of those, he asked how long a man would take to get from his house to a station a kilometre away, walking at the rate of four miles an hour. The answers varied from six weeks down to 1·1057 of a minute.

Professor Lodge had shown them a case of the graphic solution of a quadratic equation from Cremona. That had been given in Leslie's *Geometry*, where it was said to have been suggested by a student of Leslie's—*Mr. Thomas Carlyle, an ingenious young mathematician*—so that the solution shown them on the board was invented by Thomas Carlyle.

He did not think problems of construction should be entirely separated from the theorems. They should teach the student a theorem and then say to him, "Now, what good can you get from it?" They ought to let boys try things with one little difficulty on their own account. As for rapidity of computation, computation should be rapid, otherwise its value is gone; no advance would have been made over quite early times. He (the speaker) had been shown a little pamphlet which contained an Old English translation of some precepts on arithmetic that had originally been given in Latin verse. He had not troubled about the Latin verse, but he had been able to make something of the translation. Addition and subtraction were taken, and then the author goes on to "duplation," or multiplying by two. Next he takes in division by two, or "dimidiation." As an intellectual exercise this method is excellent, as they would soon be convinced if, beginning *at the right hand side* and working towards the left, they tried to divide any number by two. It will give still greater intellectual training if they divided by three or some higher number in the same way, but the British parent has a right to expect his boy to be able to calculate rapidly and well. He considered it a great thing in teaching youngsters to have figures outlined with movable rods, so as to get the idea of moving and varying quantity into them. Various little helps may be given to the pupils. You may get the idea of the three angles of a triangle being together equal to two right angles by making a paper triangle, folding in the three angles, and then you actually see they make two right angles.

[The speaker illustrated these by brown paper figures, jointed rods, etc.]

Dr. F. S. Macaulay said the invitation which had been given to Prof. Lodge was from the committee formed at the Glasgow meeting of the British Association. Although the Council of the Mathematical Association had not yet formed a committee, they had started to form one. The idea that the Council had was that they would like to get a large number of schools near London represented upon the committee, so that they might have as members those who would be able to attend the meetings. The committee would be formed mainly of those who were engaged in some form of elementary teaching.

The subject was a very difficult one. They could not hope to revolutionize the teaching of mathematics. If they could get some reforms they ought to be satisfied, and they ought not to be disappointed if their efforts did not realize what some people seemed to expect.

He, personally, was most interested in geometry. He did not altogether agree with what had been said. A document had been issued, signed by 23 masters in public schools, which had been communicated to the Committee

of the British Association, in which a good many valuable suggestions were made. The first suggestion was one on which they all seemed to be agreed :—"That the most practical direction of reform is towards a wide extension of accurate drawing and measuring." Directly we get beyond that, however, we get on to dangerous ground ; we are advised to omit such propositions of Euclid as do not "serve as landmarks." That is not clear. Again :—"We can well dispense with many propositions in the first book." It seemed to him (the speaker) that, with the exception of constructions, all the propositions in the first book were landmarks. "The third book is both easy and interesting." He agreed, but it is not so easy and interesting as the first book. The second book certainly is very difficult, and if it could be dispensed with altogether so much the better. "Euclid proves several propositions, whose truth is obvious to all but the most stupid, or the most intellectual." He did not know what this meant, for the propositions which seemed to be indicated were not those which were very difficult. They were, then, to trust to the boy himself to say whether a proposition was evident or not, and to treat him as if he were a geometer of the highest order.

He considered the fifth book of Euclid specially important because it does deal with incommensurables. He believed that the methods in the fifth book afforded the most useful and the most easy way of introducing boys to the ideas of incommensurables. The fact that incommensurables can be dealt with in geometry as well as in algebra gave them a choice as to which way should be used first. It seemed to him that the fifth book is really one of the most important parts of Euclid, if not *the* most important part.

Mr. C. Godfrey agreed as to the importance of practical geometry. At the same time riders of the ordinary type formed a valuable exercise. A greater number of boys should be able to attain to the power of doing riders. But this would always be a slow business. He, as a practical teacher, was constantly met with the impossibility of finding time. He laid the blame upon the necessity of teaching a great number of propositions. They would gain by omitting a fair number of the propositions. Restriction in the use of compasses was a restriction they were not compelled to keep up. Was not proposition I. 7 rather difficult and unnecessary? When they came to areas again, several propositions might be cut out. The use of the ordinary draughtsman's rule would get them out of several of the difficulties to which Prof. Lodge had referred. They might, too, adopt the French method of taking the third book before the second. In Book III. there were many worrying propositions. He always found a difficulty in proving that two circles cannot meet in four points. Mr. H. M. Taylor, indeed, confessed that it is impossible to draw figures which will be satisfactory in propositions of that kind. With regard to the second book, he was inclined to treat it algebraically. He admitted that he had tried it so, and had not been quite satisfied with the experiment. He found the boys' minds were not sufficiently developed at that stage, but anticipated that this difficulty would not arise if the Book II. were taken after Book III. There was a strong argument for treating the second book algebraically. In geometrical conics no one would attempt to deal a problem without algebra. One should not prohibit algebra, though the bulk of the proof might not be algebra. As for the fourth book, he had found the proofs very tedious when he was learning them. In the fifth and sixth books it is clear that simple algebraical or arithmetical treatment would meet the case of commensurables. Need they be strict about incommensurables at that stage? In their algebraic teaching, had they not made such assumptions as that $\sqrt{2} \times \sqrt{3}$ equalled $\sqrt{3} \times \sqrt{2}$? You cannot prove that unless you make some very grave assumptions at an early stage. They did not make a more serious

assumption if they confined their treatment of the sixth book to the commensurable quantities. The time for a more thorough treatment would be during the study of limits and convergent processes. Prof. Lodge's suggestion as to the constant use of squared paper he considered a very good one. In giving problems on areas a difficulty might occur as to the form in which to put the exercise. Suppose it is a trapezium whose area has to be calculated, squared paper might be very useful here. One may give the class the co-ordinates of the angular points, and from those the pupils may draw the trapezium. A graduated series of exercises can be set in this way.

Then there was the question as to the need of practising manipulation. A good teacher could give his class valuable practice orally. But if skill in manipulation is proposed as the object to be aimed at, many teachers would probably set long rows of sums from a book. He thought they should say rather less about facility of manipulation, and that the stress should be laid rather on the understanding of the process.

Mr. T. Wilson recommended the working of riders from the beginning. He would go through the first 48 propositions in the Harpur Euclid, and then proceed to one of the foreign manuals. That of Luvini, an Italian, he believed to be excellent. He did not agree with that speaker who advocated slowness in learning mathematics. He would like pupils to attain more rapidly than now some knowledge of solid geometry. He remembered he himself went right through the second book of Euclid thinking that a rectangle was a right angle. There was this to be said, however, in his favour, he used an old edition of Simson's Euclid, and in it a rectangle was nowhere defined. He did not see how they were to do without text-books. Very clever boys should have access to a good text-book: such boys would go on rapidly without a teacher's help. They all lived upon a sphere, and knew something about geography, and boys found no difficulty in understanding latitude and longitude—the co-ordinates of a sphere defining a point, and somewhat more difficult to understand than plane Cartesian co-ordinates.

He himself had been a pupil of Prof. de Morgan, and wished he had paid more attention to his instruction. De Morgan made the conception of negative quantities, the differentia between arithmetic and algebra. Boys knew the difference between north and south, and from that should have no difficulty in grasping the idea of "plus" and "minus."

Mr. T. J. Garstang confessed to being somewhat in the position of the outside assistant master. A member had expressed the wish to see some evidence of progress in the teaching of mathematics. Well, he happened to be in a school¹ where, in the absence of a governing body, the teaching was not fettered by tradition. Boys and girls were taught in the same classes together; both ladies and gentlemen were engaged to teach them. They would gather from that statement that very little attention had been paid to prejudice. He taught mathematics in this school, but he had approached mathematics from the side of science. Consequently, it seemed to him that experimental methods were obviously the right ones to adopt. When you think how easy it is for a boy to describe a circle with a pair of compasses, cut it out in cardboard, and then weigh it in a scale, you also see that the boy may learn his formula and verify it. So much can be done by such experimental methods, in the way of laying a foundation, on which one can build afterwards. He had been much influenced in his logical methods by Prof. Karl Pearson's *Grammar of Science*. Among other things this author discusses the fundamental axioms of Newton, and deals also with the metaphysical way in which the word "force" has been used. A large part of children's difficulties arises through attempts made to remember

¹[The school alluded to was Bedales School, Petersfield, Hants, recently removed from Hayward's Heath.]

definitions of words, when these definitions are not at all intelligible to them. Many of them have put down "force" as "that which communicates motion to bodies" without understanding what it means. One can get an idea of "force" sufficient for work, by treating it purely as a quantity in mathematics.

In logic, the training should go on quite gradually, and ought not to impede the children's powers of observation. He had adopted in the higher forms the text-book issued by the Association; in the lower, no text-book at all was used. From the point of view of not using a text-book, he might tell them that no text-books were used for *any* subject in the lowest forms of the school. In modern languages the children learnt to sing easy songs and to talk about common things in the modern tongue, before seeing a single word of print. He considered Prof. Chrystal's *Introduction to Algebra* far ahead of all other elementary text-books; the idea of co-ordinates is there given quite clearly. Co-ordinates occur in the first edition of Wood's *Algebra* about 1831, but drop out on revision by Lund; and it seemed to him that between 1831 and 1898 little or no progress had been made in this direction. In algebra all the work in division ought to be postponed till the boy had learnt the method of detached coefficients. Prof. Chrystal's order is much more instructive, and gets the pupil on more quickly. With regard to examinations, examiners have to try to find questions which have never been set before. They endeavoured in his school to get a boy on to some work in which he would be interested, such as electricity; he hoped to be able to teach the fundamental notions and a working knowledge of the calculus, so that a pupil might use his knowledge in the practical affairs of life when he left school. Several hundred years had passed since Cocker's *Arithmetic*, and yet the questions now set were very much like those there given. What was the good of them? Any boy well taught can do the purely arithmetical part; what he does not know in questions on stocks, etc., is, Why do shares rise or fall? and, What is the meaning of broker's commission? As part of their own work, the boys were taken out for elementary surveying. All had to go through the carpenter's shop, and through the laboratory—boy or girl. When a child had done that, there is no difficulty in teaching him decimals. He has a pair of forceps, the weights of the metric system are before him, varying in size and sometimes of different metals, and he can hardly help putting them in their proper places.

Professor Lodge, in reply, thought the use of squared paper would grow if a proper supply were always at hand.

He had been asked by Mr. Wilson to show what he meant by a hypothetical construction. Take Props. 24 and 25 of Book I. Suppose ABC , ABD are the two triangles, with common side AB , and with $AC=AD$, but the angle CAB greater than DAB . If the bisector AE of the angle CAD is drawn, cutting BC in E , we have $BC=BE+ED>BD$. The truth of this is quite independent of whether the pupil knows how to bisect an angle. If the bisector is drawn the theorem becomes evident. That is what is meant by a hypothetical construction. [*Note*.—In some of the French *Geometries* these propositions immediately follow Eucl. I. 4, with which they are evidently connected. This would have been impossible if it were necessary to elaborate a method of constructing a bisector before being entitled to use it.]

Professor Hudson had suggested that the definition of an angle should be the modern one: the rotary one. If you permit this you could prove Euclid I. 32 as early as you please. For if a line, initially coinciding with AC , revolve about A through the angle A and then about B through the angle B , the angle (BCD) which it now makes with its initial position must be equal to the sum of the angles A and B through which it has revolved. [Professor Hudson had stated that the exterior angles were shown

equal to four right angles by taking three chairs and walking once round them.]

Professor Lodge, resuming, said a reduction in the number of propositions could be made if a proposition and its converse were lumped into one, and a further simplification would be made if constructions were put in a different part of the book from theorems.

With regard to incommensurables, he considered they were not fitted for elementary subjects nor for practical work.

Rapidity of computation became interesting to boys if they were taught to emulate each other at the proper stage. He wished to call their attention to the fact that the Board of Education would issue, free, as many tables of four-figure logarithms as they liked.

He hoped the committee which was being formed would be able to make valuable suggestions to the B.A. Committee. They would be glad to receive suggestions from any members of the Association; and if these were sent to Mr. Pendlebury, no doubt he would see that they came before the committee.

THE PUBLIC SCHOOLS AND THE QUESTION.

To the Committee appointed by the British Association to Report upon the Teaching of Elementary Mathematics.

Gentlemen,—At the invitation of one of your own body, we venture to address to you some remarks on the problems with which you are dealing, from the point of view of teachers in public schools.

As regards geometry, we are of opinion that the most practical direction for reform is towards a wide extension of accurate drawing and measuring in the geometry lesson. This work is found to be easy and to interest boys; while many teachers believe that it leads to a logical habit of mind more gently and naturally than does the sudden introduction of a rigid deductive system.

It is clear that room must be found for this work by some unloading elsewhere. It may be felt convenient to retain Euclid; but perhaps the amount to be memorised might be curtailed by omitting all propositions except such as may serve for landmarks. We can well dispense with many propositions in the first book. The second book, or whatever part of it we may think essential, should be postponed till it is needed for III. 35. The third book is easy and interesting; but Euclid proves several propositions whose truth is obvious to all but the most stupid and the most intellectual. These propositions should be passed over. The fourth book is a collection of pleasant problems for geometrical drawing; and, in many cases, the proofs are tedious and uninteresting. No one teaches Book V. A serious question to be settled is—how are we to introduce proportion? Euclid's treatment is perhaps perfect. But it is clear that a simple arithmetical or algebraical explanation covers everything but the case of incommensurables. Now this case of incommensurables, though in truth the general case, is tacitly passed over in every other field of elementary work. Much of the theory of similar figures is clear to intuition. The subject provides a multitude of easy exercises in arithmetic and geometrical drawing; we run the risk of making it difficult of access by guarding the approaches with this formidable theory of proportion. We wish to suggest that Euclid's theory of proportion is properly part of higher mathematics, and that it shall not in future form part of a course of elementary geometry. To sum up our position with regard to the teaching of geometry, we are of opinion:

1. That the subject should be made arithmetical and practical by the constant use of instruments for drawing and measuring.
2. That a substantial course of such experimental work should precede any attack upon Euclid's text.
3. That a considerable number of Euclid's propositions should be omitted; and in particular
4. That the second book ought to be treated slightly, and postponed till III. 35 is reached.
5. That Euclid's treatment of proportion is unsuitable for elementary work.

Arithmetic might well be simplified by the abolition of a good many rules which are given in text-books. Elaborate exercises in vulgar fractions are dull and of doubtful utility; the same amount of time given to the use of decimals would be better spent. The contracted methods of multiplying and dividing with decimals are probably taught in most schools; when these rules are understood, there is little left to do but to apply them. Four-figure logarithms should be explained and used as soon as possible; a surprising amount of practice is needed before the pupil uses tables with confidence.

It is generally admitted that we have a duty to perform towards the metric system; this is best discharged by providing all boys with a centimetre scale, and giving them exercise in verifying geometrical propositions by measurement. Perhaps we may look forward to a time when an elementary mathematical course will include at least a term's work of such easy experiments in weighing and measuring as are now carried on in many schools under the name of Physics.

Probably it is right to teach square root as an arithmetical rule. It is unsatisfactory to deal with surds unless they can be evaluated, and the process of working out a square root to five places provides a telling introduction to a discourse on incommensurables; furthermore it is very convenient to be able to assume a knowledge of square root in teaching graphs. The same rule is needed in dealing with mean proportionals in geometry.

Cube root is harder and should be postponed until it can be studied as a particular case of Horner's method of solving equations approximately.

Passing to algebra, we find that a teacher's chief difficulty is the tendency of his pupils to use their symbols in a mechanical and unintelligent way. A boy may be able to solve equations with great readiness without having even a remote idea of the connection between the number he obtains and the equation he started from. And throughout his work he is inclined to regard algebra as a very arbitrary affair, involving the application of a number of fanciful rules to the letters of the alphabet.

If this diagnosis is accepted, we shall be led naturally to certain conclusions. It will follow that elementary work in algebra should be made to a great extent arithmetical. The pupil should be brought back continually to numerical illustrations of his work. The evaluations of complicated expressions in a , b , and c may of course become wearisome; a better way of giving this very necessary practice is by the tracing of easy graphs. Such

an exercise as plotting the graph $y = 2x - \frac{x^2}{4}$, provides a series of useful arithmetical examples, which have the advantage of being connected together in an interesting way. Subsequently, curve-tracing gives a valuable interpretation of the solutions of equations. Experience shows that this work is found to be easy and attractive.

With the desire of concentrating the attention of the pupil on the meaning rather than the form of his algebraical work, we shall be led to postpone certain branches of the subject to a somewhat later stage than is usual at present. Long division, the rule for H.C.F., literal equations, and the like,

will be studied at a period when the meaning of algebra has been sufficiently inculcated by arithmetical work. Then, and not till then, will be the time to attend to questions of algebraic form.

But at no early stage can we afford to forget the danger of relapse into mechanical work. For this reason it is much to be wished that examining bodies would agree to lay less stress upon facility of manipulation in algebra. Such facility can generally be attained by practice, but probably at the price of diminished interest and injurious economy of thought. The educational value of the subject is sacrificed to the perfecting of an instrument which in most cases is not destined for use.

To come to particulars, we think that undue weight is often given to such subjects as algebraic fractions and factors. The only types of factors which crop up continually are those of $x^2 - a^2$, $x^2 \pm 2ax + a^2$, and, generally, the quadratic function of x with numerical coefficients.

In most elementary algebra books there is a chapter on Theory of Quadratic Equations, in which a good deal of attention is paid to symmetric functions of roots of quadratics. No further use is to be made of this till the analytical theory of conics is being studied. Might not the theory of quadratics be deferred till it can be dealt with in connection with that of equations of higher degree?

Indices may be treated very slightly. The interpretation of negative and fractional indices must of course precede any attempt to produce logarithms; but when the extension of meaning is grasped, it is not necessary to spend much more time on the subject of indices; we may push on at once to the use of tables.

It will be seen that our recommendations under the head of Algebra are corollaries of two or three simple guiding thoughts; the object in view being, —to discourage mechanical work; the means suggested,—to postpone the more abstract and formal topics and, broadly speaking, to arithmetise the whole subject.

The omission of part of what is commonly taught will enable the pupil to study, concurrently with Euclid VI., a certain type of diluted trigonometry which is found to be within the power of every sensible boy. He will be told what is the meaning of sine, cosine, and tangent of an acute angle, and will be set to calculate these functions for a few angles by drawing and measurement. He will then be shown where to find the functions tabulated, and his subsequent work for that term will consist largely in the use of instruments, tables, and common-sense. A considerable choice of problems is available at once. He may solve right-angled triangles, work sums on "heights and distances," plot the graphs of functions of angles, and make some progress in the general solution of triangles by dividing the triangle into right-angled triangles. Only two trigonometrical identities should be introduced— $\sin^2 \theta + \cos^2 \theta = 1$, and $\frac{\sin \theta}{\cos \theta} = \tan \theta$. In short, the work should be arithmetic, and not algebra.

Formal algebra cannot be postponed indefinitely; perhaps now will be the time to return to that neglected science. We might introduce here a revision course of algebra, bringing in literal equations, irrational equations, and simultaneous quadratics, illustrated by graphs, partial fractions, and binomial theorem for positive integral index. Side by side with this it ought to be possible to do some easy work in mechanics. Graphical statics may be made very simple; if it is taken up at this stage, it might be well to begin with an experimental verification of the parallelogram of forces, though some teachers prefer to follow the historical order and start from machines and parallel forces. Dynamics is rather more abstract; a first course ought probably to be confined to the dynamics of rectilinear motion.

It is not necessary to discuss any later developments. The plan we have

advocated will have the advantage of bringing the pupil at a comparatively early stage within view of the elements of new subjects. Even if this is effected at the sacrifice of some deftness in handling a , b , and c , one may hope that the gain in interest will be a motive power of sufficient strength to carry the student over the drudgery at a later stage. Some drudgery is inevitable if he is ultimately to make any use of mathematics. But it must be borne in mind that this will not be required of the great majority of boys at a public school.—We beg to remain, Gentlemen, yours faithfully,

G. M. BELL, Winchester; H. H. CHAMPION, Uppingham; H. CRABTREE, Charterhouse; F. W. DOBBS, Eton; C. GODFREY, Winchester; H. T. HOLMES, Merchant Taylors' School; G. H. J. HURST, Eton; C. H. JONES, Uppingham; H. H. KEMBLE, Charterhouse; T. KENSINGTON, Winchester; E. M. LANGLEY, Bedford Modern School; R. LEVETT, King Edward's School, Birmingham; J. W. MARSHALL, Charterhouse; L. MARSHALL, Charterhouse; C. W. PAYNE, Merchant Taylors' School; E. A. PRICE, Winchester; D. S. SHORTO, Rugby; A. W. SIDDON, Harrow; R. C. SLATER, Charterhouse; H. C. STEEL, Winchester; C. O. TUCKER, Charterhouse; F. J. WHIPPLE, Merchant Taylors' School.

REVIEWS AND NOTICES.

Choice and Chance, with 1000 Exercises. By PREBENDARY W. A. WHITWORTH. Fifth Edition, much enlarged. Pp. viii, 342. 6s. 1901. (Deighton, Bell.)

This interesting volume "contains some 45 pages more than its predecessor." "The most important addition in the body of the work is the very far-reaching theorem which enables us to write down at sight the value of such functions as $a^2, a^2\beta^4, a\beta\gamma, \dots$ when $a, \beta, \gamma, \dots$ are the parts into which a given magnitude is divided at random." The exercises at the end are increased to 1000. Prebendary Whitworth's work is too well known to require further comment.

Algebra for Junior Students. By TELFORD VARLEY. Pp. viii, 152. 1s. 1901. (Allman.)

This little book covers "all examinations intended for younger students." It is thoughtfully written and better than most of its class. It is a pity that detached coefficients are not taught as early as possible, and that the solution of quadratics by splitting into factors is not mentioned. When the author states in his preface that the contents may be mastered in two years, we despair of mathematical teaching in this country and heartily sympathise with Professor Perry. There is no reason why the contents should not be mastered in two terms at most.

Elementary Algebra. By ROBERT GRAHAM. Third Edition. Pp. viii, 312 (34). 6s. 1901. (Longmans, Green.)

This edition contains some 28 pages more than in the first edition. The best chapters in this book are those dealing with imaginary quantities, imaginary factors, and "more difficult equations." The author does not teach the use of detached coefficients, but he is "advanced" enough to use determinants in certain equations of three unknowns, though we cannot see why he did not introduce them to the student who is tackling equations with two unknowns. The examples are very carefully graduated.

Woolwich Mathematical Papers for Admission into the Royal Military Academy for the years 1891-1900. Edited by E. J. BROOKSMITH. 6s. 1901. (Macmillan.)

This volume will be found serviceable to Army Classes. It contains the questions in Pure and Applied Mathematics for the years mentioned, together with the answers. Mr. Brooksmith is an Instructor at the Royal Military Academy.

The Tutorial Algebra. Part I, Elementary Course. By R. DEAKIN. Pp. viii., 443. 1901. 3s. 6d. (University Tutorial Press.)

"Suppose $+2 \times -3$ or $-2 \times +3$. Evidently the product will not be the same in either of these cases as in $+2 \times +3$. Therefore we conclude that $+2 \times -3 = -6$ and $-2 \times +3 = -6$... Again suppose -2×-3 . This is different from the last two cases, and we conclude that $-2 \times -3 = +6$... From these results we can infer the rule of signs." The writer claims in his preface that he has tried to encourage thought from the very beginning of the work. The quotation is a specimen of the way he does it—on p. 26. We do not see that this book is any better than most of its rivals, but for the inclusion of a chapter on graphs. It is well printed and got up.

Practical Mathematics for Beginners. By F. CASTLE. Pp. ix., 314. 2s. 6d. 1901. (Macmillan.)

Elementary Practical Mathematics. By M. T. ORMSBY. Pp. xii., 410. 1900. (Spon.)

These books, of course, owe their existence to the inspiration of Prof. Perry, whose crusade against the present system of teaching Mathematics in our schools is likely to be effectual. Mr. Castle's larger volume on the same subject has already met with a considerable measure of success. The book under notice is designed to cover the ground laid down in the syllabus recently issued by the Board of Education. Within those lines it is excellent. Mr. Ormsby, while also endeavouring to meet the requirements of the same Board, has had in view the needs of more advanced students, especially students of Civil Engineering, who wish to be taught "the additional matter they are likely to require at an early stage, and how to use their knowledge, when acquired, for practical calculations." Mr. Ormsby is a good teacher, as may be seen from the careful way in which most of the demonstrations are reasoned out for the benefit of the private student.

Mathematisches Vokabularium. By F. MÜLLER. Französisch-Deutsch und Deutsch-Französisch. Zweite Hälfte. Pp. xiv., 316. 1901. 11 m. (Teubner, Leipzig.)

We have already drawn the attention of readers of the *Gazette* to the first part of this dictionary. On page xi. of the second part appear some sixty "Nachträge" to the first part, and it does credit to Herr Müller that but three of these are corrections. On the other hand there are some two hundred additions and corrections to the second part on pp. xii.-xiv., and of these a large proportion are "Verbesserungen." It should, however, be noticed that there are fifty more pages in the second part than in its predecessor. This is due to the fact that there are fewer technical terms in French than in German, and also to an increase in the total number of additions and historical references. There are still a few misprints of an obvious nature, such as "proportionale," p. 213. A careful study of selected portions of the text confirms us in the favourable impression produced upon our mind by the first part. Herr Müller reminds us of the saying attributed to the great Littré, "les travaux lexicographiques n'ont point de fin." That no doubt is true, but anyone who has used this dictionary for, say five years, by the extent of reading that implies, will find himself practically independent of such additional matter as it may become necessary to incorporate into the volume during an ordinary lifetime. To the young student this dictionary should prove invaluable.

BOOKS, ETC., RECEIVED.

Annals of Mathematics. Edited by ORMOND STONE and others. Second Series. Vol. III. No. 1. Oct. 1901. pp. 44. 2/- (Longmans, Green).

[On the Convergence of the Continued Fraction of Gauss and Other Continued Fractions: Van Vleck. On the Differentiation of an Infinite Series Term by Term: Forster. A note on Geodesic Circles: Whittemore. Note on certain Functions defined by Infinite Series: Osgood. Nim, a game with a complete Mathematical Theory: Bouton. On the Groups generated by Two Operators of Order Three, whose product is also of Order Three: Miller. On the Invariants of a Quadrangle under the largest sub-group, having a fixed point, of the General Projective Group in the Plane: Granville.]

Annals of Mathematics. Edited by ORMOND STONE and others. 2nd Series, III. 2. (Harvard University and Longmans.) Jan., 1902. 2/-.

[Some applications of the method of Abridged Notation: Böcher. On the roots of functions connected by a Linear Recurrent Relation of the Second Order: Porter. Spaces of Constant Curvature: Woods.]

Leçons d'Algèbre. By MME. A. SALOMON. pp. 184. 2 fr. 1901 (Nony).

Woolwich Mathematical Papers, 1891-1900. Edited by E. J. BROOKSMITH. 6s. 1901 (Macmillan).

Practical Mathematics for Beginners. By F. CASTLE. pp. ix., 314. 2/6. 1901 (Macmillan).

Text-Book of Practical Solid Geometry, etc. By Capt. E. H. de V. ATKINSON, R.E. 2nd Edition. pp. 124. 7/6. 1901 (Spon).

Nautical Astronomy. By J. H. COLVIN. pp. 127. 2/6 net. 1901 (Spon).

An Elementary Treatise on the Calculus. By G. A. GIBSON. pp. xix., 459. 1901 (Macmillan).

Annuaire pour l'an 1902. Publié par le Bureau des Longitudes. pp. 656 + 185. 1 fr. 50 c. 1902 (Gauthier-Villars).

Geometric Exercises in Paper Folding. By T. SUNDARA ROW. pp. x., 148. 4/6 net. 1901 (Kegan Paul, Open Court).

Culegere de Probleme de Arithmetica, Geometrie, Algebra, și Trigonometrie. Edited by Messrs. IONESCU, and ȚIȚICA. pp. viii., 520. 1901 (Göbl, Bucharest).

Spherical Trigonometry. By the late I. TODHUNTER. Revised by J. G. LEATHAM. pp. ix., 275. 7/6. 1901 (Macmillan).

**Grundrissen des Politischen Arithmetik.* By M. KITT. pp. 78 + pp. 39 (tables). 3 marks. 1901 (Teubner).

**Lehrbuch der Combinatorik.* By E. NETTO. pp. 258. 9 marks. 1901 (Teubner).

**Der Naturwissenschaftliche Unterricht in England insbesondere in Physik und Chemie.* By K. T. FISCHER. pp. 94. 3 m. 60 pf. 1901 (Teubner).

**Theorie der Riemann'schen Thetafunktion.* By Dr. ROST. pp. 66. 1901 (Teubner).

**Jahresbericht der Deutschen Math.-Verein.* x Band. 2 Heft. 1 Hälfte. Burkhardt. *Entwickelungen nach oscillierenden Funktionen.* pp. 176. 5m. 60. 1901 (Teubner).

Leçons sur les Séries à Termes Positifs. By EMILE BOREL. pp. 94. 1902 (Gauthier-Villars).

La Géométrie Atomique Rationnelle. By J. F. BONNELL. pp. 100. 1902 (Gauthier-Villars).

Theoretical Mechanics. By W. WOOLSEY JOHNSON. p. 434. 1901 (Wiley, Chapman Hall).

Traité de Cinématique Théorique. By H. SICARD, annotated by A. LABROUSSE. pp. 187. 1902 (Gauthier-Villars).

Cours de Mécanique. By P. APPELL. pp. 272. 1902 (Gauthier-Villars).

College Algebra. By J. H. BOYD. 21 + 11 + 787 pp. 1901. Hf. leather. \$2. (Scott & Foresman, Chicago.)

Aufgaben zur Differential- und Integralrechnung. By H. DÜLP and E. NETTO. 3 + 216 pp. Cloth. 4 m. 1901 (Ricker, Giessen).

Supplementary Report on Non-Euclidean Geometry. By G. B. HALSTED. [Science, N.S. 14; pp. 705-717.]

Linear Groups, with an Exposition of the Galois Field Theory. By L. E. DIXON. 10 + 312 pp. Cloth. 12 m. 1901 (Teubner, Leipzig).

Bulletin of the American Mathematical Society. Edited by F. MORLEY and others. Series 2. Vol. VIII. No. 3. Dec. 1901.

[Modern Methods of Treating Dynamical Problems: E. W. Brown. Some Curious Properties of Conics touching the Line Infinity at one of the Circular Points: E. F. Huntington and J. K. Whittemore.]

Ditto. No. 4. Jan. 1902.

[Note on Mr. G. Pierce's Approximate Construction for π : E. Lemoine. Concerning the $P(g, g, 2)$ functions as coordinates in a line complex: H. F. Stecker. Abelian Groups conformal with non-Abelian Groups: G. A. Miller. The Infinitesimal Generators of certain Parameter Groups: S. E. Slocum.]

MACMILLAN & CO.'S

Latest Mathematical Works.

BRITISH ASSOCIATION MEETING AT GLASGOW, 1901.

Discussion on the Teaching of Mathematics, which took place on September 14 at a Joint Meeting of two Sections. Section A.—Mathematics and Physics. Section L.—Education. Chairman of the Joint Meeting, the Right Hon. Sir John E. Gorst, K.C., M.P. Edited by JOHN PERRY. Crown 8vo, stiff boards, 2s. net.
Academy.—"A unique collection of competent opinion on this important subject."

An Elementary Treatise on the Calculus. With Illustrations from Geometry, Mechanics, and Physics. By Professor GEORGE A. GIBSON, M.A., F.R.S.E. Crown 8vo, 7s. 6d.

Educational News.—"Teachers of higher Mathematics would do well to make a careful examination of this book. Many members of our profession have, through disuse, allowed their knowledge of the Calculus to rust; to those who can recommend no better book than this, for the Author is most up to date in his facts and nomenclature."

Primer of Geometry. Comprising the Subject Matter of Euclid I.-IV., treated by the Methods of Pure Geometry. By H. W. CROOME SMITH, B.A. Crown 8vo, 2s.

Royal College of Science Magazine.—"Will speedily find favour, especially with teachers and students who have already found a want of common sense in the ordinary method of treating geometry."

Spherical Trigonometry for the use of Colleges and Schools. By the late I. TODDUNTER, M.A., F.R.S. Revised by J. G. LEATHER, M.A., D.Sc., Fellow and Lecturer of St. John's College, Cambridge. Crown 8vo, 7s. 6d.

Practical Mathematics for Beginners. By FRANK CASTLE, M.I.M.E., Mechanical Laboratory, Royal College of Science, S. Kensington. Globe 8vo, 2s. 6d.

Euclid's Elements of Geometry. By CHARLES SMITH, M.A., and SOPHIE BRYANT, D.Sc. Globe 8vo. Book I., 1s. Books I. and II., 1s. 6d. Books III. and IV., 2s. Books I.-IV., 3s. Books I.-IV., VI., and XI., 4s. 6d. Books VI. and XI., 1s. 6d.

Journal of Education.—"The whole work is now before us, and we have no hesitation in saying that it occupies a place in the very front rank of English text-books of Geometry."

Elementary Geometry, Plane and Solid, for Use in High Schools and Academies. By Professor THOMAS F. HOLGATE. Crown 8vo, half leather, 6s.

Quarterly.—"The treatment of the subject is sound, the divisions and sub-divisions are natural, and the correlation of the parts to the whole is particularly satisfactory. . . . A thoroughly satisfactory exposition of the author's aims."

Algebraical Examples Supplementary to HALL and KNIGHT'S ALGEBRA FOR BEGINNERS and ELEMENTARY ALGEBRA (Chaps. I.-XXVII.) By H. S. HALL, M.A. With, or without Answers. Globe 8vo, 2s.

MACMILLAN & CO., LIMITED, LONDON.

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY.

Vol. XIX., pp. 4, 77.

Session 1900-1901 (7/6). Williams and Norgate.

CONTENTS:—Allardice, R. E., *On four circles touching a common circle. On the nine-point conic. On a cubic curve connected with the triangle*; Carslaw, H. S., *Oblique Incidence of a Train of Plane Waves on a semi-infinite Plane*; Chrystal, G., *Elementary Theorems on Surds*; Davis, R. F., *Focal Relations of a Bicircular Quartic*; Gibson, G. A., *An extension of Abel's Theorem on the continuity of a Power Series*; Jack, J., *Change of Area. Director Circle*; Mair, D., *On the n^{th} root of a prime*; Muirhead, R. F., *Inequalities*; Third, J. A., *Triangles triply in perspective*; Tweedie, C., *Area of Triangle in Cartesian. Addition Theorem in Trigonometry.*

THE MATHEMATICAL ASSOCIATION.

SPECIAL NOTICE.

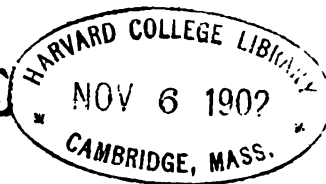
At a Meeting of the Council held on February 4, 1902, the present financial position of the Association was very carefully considered.

At the beginning of 1901 the balance, after deducting liabilities, was about £16. At the end of the year there was a deficit of about £1. Moreover, the difference of £17 represents only part of the actual loss during 1901, owing to the exceptional nature of certain receipts. The loss is a consequence of the increased cost of the "Mathematical Gazette." The Council believes that the decision to double the number of issues during 1901 has had the unanimous approval of the members of the Association, and is of opinion that any enforced diminution in the present issue of six numbers a year would be detrimental to the best interests of the Association.

At the beginning of 1895, the Association had a sufficient balance in hand to justify the reduction of the annual subscription to 7s. 6d. Since that time, however, the work and expenses in connection with the "Gazette" have steadily increased; and the balance, notwithstanding the strictest economy, has disappeared. The Council has therefore decided to raise the subscription, commencing with the year 1903, to its original amount of 10s. It is hoped that every member of the Association will assist the efforts of the Council to bring about a more prosperous state of finance. To effect this it is necessary not only that there should be no loss in membership, but that the present substantial rate of increase in numbers should be maintained.

The published price of the "Gazette" is now raised to 1s. 6d. net.; but members of the Association may purchase extra copies from the Secretaries at the price of 1s. each.

THE MATHEMATICAL GAZETTE



EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 35.

OCTOBER, 1902.

1s. 6d. Net.

The attention of Members and Subscribers is drawn to the Special
Notice on the back cover.

The Mathematical Gazette is issued in January, March, May, July,
October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, Alford House, Aberdeen.

All other correspondence should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

NOTICE.—The Index and Title-page to Volume I. can be obtained from the Publishers.

CONTENTS.

	PAGE
GENERAL THEORY OF VERNIERS. II. P. J. HEAWOOD, M.A., - - -	237
GEOMETRY IN FLATLAND. Prof. A. C. DIXON, M.A., - - -	241
MATHEMATICAL NOTE. The Rev. Prebendary W. A. WHITWORTH, M.A., -	242
REVIEWS. J. E. WRIGHT, M.A.; R. W. H. T. HUDSON, M.A.; A. W. SIDDONS, M.A., - - -	243
PROBLEMS, - - -	250
SOLUTIONS, - - -	251
BOOKS, ETC., RECEIVED, - - -	252
ERRATA, - - -	252

Forthcoming Articles and Reviews by:—

Prof. A. C. DIXON; Prof. T. J. I'AN BROMWICH; Prof. G. B. MATHEWS;
E. T. WHITTAKER, M.A.; R. W. H. T. HUDSON, M.A.; W. N. ROSE-
VEARE, M.A.; P. J. HEAWOOD, M.A.; J. H. GRACE, M.A.; and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. *No. 8 is out of print.* A few numbers may be obtained from the *Editor* at the following prices, post free:

No. 7, 5s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 15s.

These will soon be out of print and very scarce.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied, except on payment of 1s. per copy.

THE MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

JANUARY, 1903.

No. 37.

GENERAL THEORY OF VERNIERS.

II.

3. We have seen the connection between the general theory of verniers and the indeterminate equation of the form $pn - qm = r$; the equation $pn + qm = r$ might be illustrated in much the same way. But, taking the case originally considered, we observe that it is the same question as before, from a different point of view, if instead of starting with an interval AB between divisions of two contiguous scales, and asking how far we must go to reach a coincidence, we suppose the zeros of two scales to coincide and ask how far we have to go to reach a certain interval, between divisions of the two. The question only differs in so far as it suggests the simultaneous consideration of more than one such interval. But further if, instead of repeated divisions of each kind, we take a circle whose entire circumference = one of the larger divisions, a say, and from a fixed point A on the circumference measure off continually arcs = to the smaller division b , (figs. 5, 6), the question how far we have to go to reach a certain approximation to the starting point is the same thing in another form, although the arrangement brings to light many new relations, which in their turn throw light on the result. We have the subdivisions of the different a -intervals superposed as it were upon one a -interval, and the circle is thus continually divided into more and more parts until (if a and b are commensurable) exact coincidence with the starting point is reached. But while the distance x of the p th point of division from A the starting point, (supposing p such intervals to involve q circuits), is defined by the same equation $pb \sim qa = x$ as the distance involved in the vernier problem, there are important results, which did not before present themselves,

founded on the consideration of the way in which the circumference is continually split off into smaller portions as the process of division goes on, by points which, though not consecutive in the order in which they are taken, come near together on the circumference of the circle. These results depend on the obvious principle that if the k th point of division comes at a certain distance from the k th, the $h + l$ th will come at the same distance from the $k + l$ th, (whatever these numbers may be). In particular no interval can occur between any two points, which has not already occurred between some point and A . So again if, after inserting p points, we reach a point at a distance x from A , the next p points will occur at distances x , respectively, from the first p . Thus when, in our first circuit, we reach a point B , (fig. 5), at an interval c , ($< b$) from A , (supposing $a = mb + c$), the next m points will occur at the same distance c , respectively,

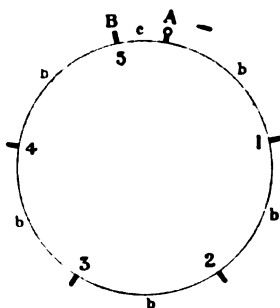


FIG. 5.

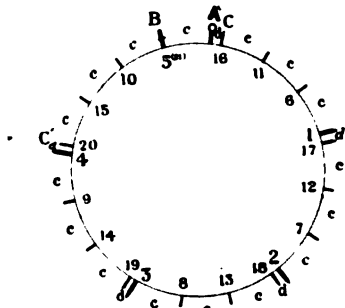


FIG. 6.

from the first m , and so on. Supposing $b = m'c + d$, ($d < c$), m' circuits will be thus spent in cutting off successive portions c from the b -divisions, until each is divided into m' parts $= c$, and one $= d$. When C (fig. 6) is thus reached, we have the whole circumference divided into parts of lengths c, d ; and the numbers of these parts being $mm' + 1$ and m , or p' and p say;

$$a = (mm' + 1)c + md = p'c + pd.$$

But the first of these d -segments will be adjacent to A , being that between A and C , where C is the p th point from A . Then and not till then shall we reach a point nearer to A than B was. If q' be the corresponding number of circuits, ($= m' + 1$), $p'b - q'a = d$, (as indeed $qa - pb = c$, where $q = 1$).

It is not difficult to see that after the further stage shown in fig. 6 is reached, as described above, a like process will be repeated. In the figure $m = 5$, $m' = 3$, and, dealing with actual numbers, we have there, at the insertion of the 20th point of division, 16 c -intervals and 5 d -intervals; and the next 16 points

will cut off a length d from each c , and so on, (for the 20th point being at that distance from the 4th the 21st will come at the same distance from the 5th, etc., etc.). Suppose that $c = 4d + e$, ($e < d$). Then the insertion of 4×16 or 64 points after C , (bringing us up to No. 84), will leave the circumference divided into $64 + 5$, or 69 points each $= d$, with 16 each $= e$. But the first of the e -divisions will occur next to A , (on the side of B), when the 69th point is introduced; while it is obvious that until then we shall never again be so near to A , as we were at C . After No. 84 we shall have a similar cycle of subdivision, ending when the circumference is completely divided into parts equal either to e or to the next smaller division, and so on continually. Segments of three different lengths will occur at intermediate stages.

The general law involved is not difficult to trace. Perhaps the regularity of the process is least obvious, when we come to one of the series $m, m', \dots$, which $= 1$. If *e.g.* $m' = 1$, or the division c (introduced with the fifth point in fig. 5), were $> \frac{1}{2}b$, the next smaller division d would be introduced by the insertion of the very next point; but the cycle would be still quite regular, only somewhat compressed, reducing simply to its final stage, corresponding to that from C to C' in the actual case. Using suffixes, the general result may be formulated as follows. After reaching a stage when the circumference is divided into p_r parts each $= a_{r-1}$ and p_{r-1} parts each $= a_r$, (so that we have $p_r a_{r-1} + p_{r-1} a_r = a$), the succeeding points of division will cut off from each of the larger segments in succession a part equal to the smaller, and then from each another like part again, if the larger is more than twice the smaller segment, and so on; and not until we begin the cutting off of the last such part, (which will happen first with the segment next to A), shall we approach so near to A again as we have already been. When the end of this cycle is reached, the circumference will be completely divided into portions of lengths a_r and a_{r+1} ($< a_r$): it is not difficult to see that the first introduction of such a new smallest segment will be alternately in advance of A and behind it, for successive values of r . Supposing

$$a_{r-1} = m_{r+1} a_r + a_{r+1}, \dots \dots \dots (1)$$

there will be $m_{r+1} p_r$ points in the complete cycle we are considering. Since each introduces a new division a_r , and we start with p_{r-1} such, the number of these at its close

$$\equiv p_{r+1} = m_{r+1} p_r + p_{r-1}, \dots \dots \dots (2)$$

so that, with p_r of the smaller divisions left as remainders from the old a_{r-1} -divisions, we have

$$p_{r+1} a_r + p_r a_{r+1} = a. \dots \dots \dots (3)$$

Moreover the first of the a_{r+1} -segments will be that introduced between the p_{r+1} th point and A , so that we have finally

$$p_{r+1}b - q_{r+1}a = (-1)^{r+1}a_{r+1}, \dots \dots \dots (4)$$

with a proper value of q_{r+1} , (the sign alternating as pointed out above); and it is not difficult to see that the q 's follow the same law of formation as the p 's. From the stage defined by (3) we proceed as before. If a, b are commensurable, we shall finally have the entire circumference divided into parts equal to the G.C.M. of a and b .

4. Now these results illustrate graphically, indeed they may be said to amount to a graphical proof of, the fundamental results in the theory of the convergents to a continued fraction. The graphical result we obtained, that after a new shortest division next to A is introduced with the p_r th point, we cannot again approach so near to A until the p_{r+1} th point, as there defined, is reached (with the numerical results which follow), involves the fact that for values of p_r determined in succession by means of equations (1), (2) from obvious initial values, we have $p_r b - q_r a = a_r$, a quantity smaller than is possible for any other value than p_r until p_{r+1} is reached; which is the primary result in question.

By means of equation (3) we can deduce other elementary relations. Thus, taking (4) for two consecutive values of r :

$$p_r b - q_r a = (-1)^r a_r,$$

$$p_{r+1} b - q_{r+1} a = (-1)^{r+1} a_{r+1},$$

we deduce

$$(p_r q_{r+1} - p_{r+1} q_r) a = (-1)^r (p_{r+1} a_r + p_r a_{r+1}) = (-1)^r a, \text{ by (3),}$$

or

$$p_r q_{r+1} - p_{r+1} q_r = (-1)^r. \dots \dots \dots (5)$$

Hence again, by means of the same pair of equations,

$$q_{r+1} a_r + q_r a_{r+1} = (-1)^r (p_r q_{r+1} - p_{r+1} q_r) b = b, \dots \dots \dots (6)$$

a fact which may also be verified graphically. Finally, if (4) be taken for three successive values of the suffixes $r-1, r, r+1$, we have by means of (1), (2) the result

$$q_{r+1} = m_{r+1} q_r + q_{r-1}, \dots \dots \dots (7)$$

although this may also be inferred otherwise. Other points of interest might be mentioned, such as the consideration of the divisions answering to "intermediate convergents," but the above are the results which follow most naturally from the graphical point of view.

PERCY J. HEAWOOD.

DURHAM, October, 1902.

GEOMETRY IN FLATLAND.

AN inhabitant of Flatland would not be in the same position with regard to elementary geometry as a three-dimensioned being. He could only use superposition for figures of the same aspect. I have not met with any discussion of the way in which he would prove, say, Euc. I. 5; the following seems complicated, but at least it shews that the scope of the usual assumption as to superposition can be made narrower.

The list of theorems that do not depend on the case of superposition in which a figure has to be taken out of its plane includes the following—cases of I. 4 and 26, I. 13-17, 27-30, 32-41, 43, 47, and the theorems of Book II. The following are excluded—cases of I. 4 and 26, I. 5-8, 18-21, 24, 25, 48—we are to prove them by other means.

The constructions of I. 9, 11, 12, 46 are not available, but those of I. 1, 2, 3, 10, 22, 23, 31, 42, 44, 45 may be used. Though we cannot bisect an angle, it is clear that any angle must have a bisector, and only one. The following may be given as a proof:

If the angle BAC has no bisector, any line drawn through A between AB and AC divides it into unequal parts. Suppose AX to be that line, for which $\hat{BAX}$ exceeds $\hat{XAC}$ but by the least possible difference, and AY that for which $\hat{YAC}$ exceeds $\hat{BAY}$ but by the least possible difference.

Then twice $\hat{BAX} > \hat{BAX} + \hat{XAC} > \hat{BAC}$,

and twice $\hat{BAY} < \hat{BAY} + \hat{YAC} < \hat{BAC}$,

so that

$\hat{BAY} < \hat{BAX}$.

Draw a line AZ between AX and AY , then if $\hat{BAZ} > \hat{ZAC}$ the excess $< \hat{BAX} - \hat{XAC}$, and if $\hat{BAZ} < \hat{ZAC}$ the defect $< \hat{YAC} - \hat{BAY}$, which is contrary to the supposition.

Hence the angle BAC must have a bisector.

The fundamental assumption in this proof is that any angle, however small, can be divided into parts.

By taking BAC to be a flat angle we have as a corollary that at any point in any straight line there is a perpendicular.

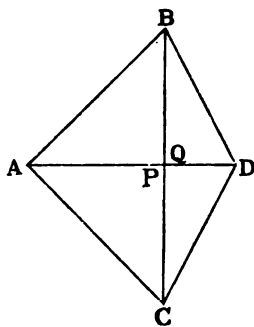
Through any external point a parallel to this passes. Thus there is one perpendicular (and only one) to any given straight line from any point, whether in it or outside it.

Now let AB be any finite straight line, AD , BC perpendiculars to it at A , B whose lengths are equal to AB . The figure $ABCD$ is a parallelogram (I. 33), and in fact a square. On any straight line there are two squares, one on each side, but these are equal in area (I. 36), and thus the "area of the square on a straight line" is an unambiguous expression. Similarly for a rectangle.

Thus though we may not construct perpendiculars and squares, we may, for the purpose of proving theorems, suppose them drawn, since they certainly exist.

(A) Let ABC be a triangle in which $AB=AC$. Suppose AD to be the bisector of the angle A . Then shall AD bisect BC at right angles.

Let BP , CQ be the perpendiculars from B , C to AD . Cut off $AD=AB$ or AC .



The triangles BAD , DAC are congruent, so that $BD=DC$.

$$\begin{aligned}\text{Hence} \quad 2AP \cdot AD &= AB^2 + AD^2 - BD^2 & (\text{II. 12 or 13}) \\ &= AC^2 + AD^2 - CD^2 \\ &= 2AQ \cdot AD.\end{aligned}$$

$$\text{Hence} \quad AP=AQ \quad (\text{I. 34 and 40}).$$

$$\text{Hence} \quad BP=QC \quad (\text{I. 47}).$$

Since $AP=AQ$, P and Q coincide, and thus the proposition is proved.

It follows that the perpendicular from A to BC bisects the angle A and the line BC , and that the median from A bisects the angle A and is perpendicular to BC .

(B) Now let EFG , HFG be two triangles on opposite sides of FG , having $EF=HF$ and the angles EFG , GFG equal. Then shall the triangles be equal in all respects.

Draw EH cutting FG , produced if need be, in R . By (A), FG bisects EH at right angles.

$$\text{Hence} \quad EG=HG \quad (\text{I. 47}).$$

Since EGH is isosceles GR bisects the angle EGH . Hence the angles EGF , FGR are equal.

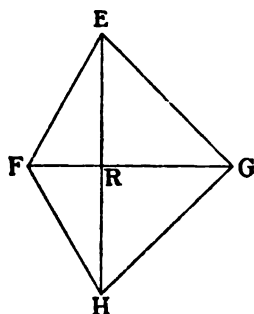
The third angles are equal (I. 32) and the areas (I. 38).

The second (symmetrical) case of I. 4, and I. 5, follow at once from this. I. 5 could also be deduced from (A) and I. 32.

For I. 8 in the symmetrical case put the bases together and use I. 5. In the congruent case compare the triangles with a symmetrical one.

In the symmetrical case of I. 26, which includes I. 6, compare the triangles with a third, made symmetrical with one of them by I. 23 and 3.

A. C. DIXON.



MATHEMATICAL NOTE.

119. [I. 13. b. a; 17. c.]

Rule for finding the number of quarts not greater than a given number N (a quart being a number which cannot be expressed as the sum of one, two, or three squares).

1. Express N in the binary scale, and let its digits, commencing with the unit-digit, be a_0, a_1, a_2, a_3 , etc. (Of course each a is either 0 or 1.)

2. In this expression for N note sequences of three 1's; but only count such sequences of three as end with the unit-digit, or with a digit an *even* number of places from the unit-digit. Let β be the number of these sequences.

3. Then the number of quarts not exceeding N will be

$$\begin{aligned}\beta + (a_3 + a_6 + a_7 + \dots) + 2(a_4 + a_6 + \dots) + 4(a_5 + a_7 + \dots) \\ + 8(a_6 + a_8 + \dots) + \text{etc.}\end{aligned}$$

Examples of the Rule.

Ex. 1. To find the number of quarts not exceeding 125.

$125 = 1111101$ binary, and there are two sequences as marked. Hence the number of quarts is

$$2 + (1+1) + 2(1+1) + 4 \cdot 1 + 8 \cdot 1 = 20.$$

Ex. 2. To find the number of quarts not exceeding 167.

167 = 10100111 binary, and there is one sequence as marked. Hence the number of quarts is

$$1 + (0+1+1) + 2(0+0) + 4(1+1) + 8 \cdot 0 + 16 \cdot 1 = 27.$$

N.B.—These 27 quarts are—7, 15, 23, 28, 31, 39, 47, 55, 60, 63, 71, 79, 87, 92, 95, 103, 111, 112, 119, 124, 127, 135, 143, 151, 156, 159, 167.

W. A. WHITWORTH.

REVIEWS.

Cours d'Analyse Mathématique. Tome I. By É. GOURSAT. Pp. 620. 1902. (20fr.)

This work is practically the résumé of a course of lectures given by the author to the Faculté des Sciences de Paris. The attempt has been made to give in this the first volume a general exposition of the properties of functions of real variables, the exception being made of those connected with differential equations. This part of the subject, however, will be treated in a second volume at present in the press.

As was perhaps to be expected from M. Goursat, the book is one of the most pleasing which has appeared recently on the subject in question. The matter is treated in a vigorous manner, and its arrangement leaves little to be desired. The author assumes that the student has some acquaintance with the elements of the calculus, and he also assumes some knowledge of the better known properties of irrationals. Whilst admitting that the theory of irrationals ought logically to form the ground-work of an exposition of Mathematical Analysis, appeal is made to the many well-known works on the subject.

The contents of the volume are divided primarily into four parts—(1) General theorems on differentiation, (2) Integration and properties of Definite Integrals, (3) Theory of Series, and (4) Geometrical Applications.

The first part commences with theorems on continuity and on limits. General theorems connected with differentiation are then given, some space being devoted to a consideration of the properties of Jacobians and of Hessians, and to a discussion of various transformations, such as those of contact, for example.

Taylor's Theorem, and its extension to several variables, together with properties of functions connected with it, form the next chapter, while the remainder of this part is devoted to a discussion of maxima and minima, and to problems connected therewith.

The second part commences with a close examination into the question of continuity from an analytical standpoint. General theorems on integration are then given, which are afterwards exemplified by appeal to geometrical intuition. After a discussion of ordinary methods of integration the author proceeds to consider double and multiple integrals, some space being devoted to Green's and Stokes' theorems. A few interesting examples of definite integrals are given, and this part concludes with a short account of the integration of total differentials.

The next two chapters are concerned with the theory of series. The first of these is occupied with general convergence criteria. For the purpose of obtaining many of them, use is made of Cauchy's theorem that

$$\sum_{x=0}^{\infty} \phi(a+x) \text{ and } \int_a^{\infty} \phi(x)dx$$

converge or diverge together.

There are also given in this part of the work sections on multiple series and on series with variable terms. The section on properties of power series is particularly worthy of note. The division concludes with a discussion of trigonometric series, the proof of Fourier's theorem given being Bonnet's modification of that due to Lejeune-Dirichlet.

The remainder of the volume is occupied with general properties of plane and twisted curves and of surfaces.

An interesting feature of the work as a whole is the number of examples given, both solved and unsolved. Those solved seem particularly adapted to illustrate the points desired, and the exercises given at the ends of the various chapters form a collection which is sure to be extremely useful.

Altogether the author is to be congratulated on having produced a very interesting elementary treatise on Mathematical Analysis, and one which is sure to be found widely useful. J. E. WRIGHT.

Elementary Geometry. W. M. BAKER and A. A. BOURNE. Pp. viii. + 211. 2s. 6d. Second Edition revised. 1902. (Bell.)

The appearance of this and similar books marks an epoch in the history of geometry. The time seems to have come when Euclid, considered as an introduction to geometrical ideas, must at last be abandoned in this country. Even as a system of logic he falls before the searching requirements of the modern founders of geometry, and it would appear that the pre-eminent place he has occupied for over two thousand years must be vacated. It cannot be without a feeling of sentimental regret that we watch him being relegated to antiquity while our better judgment bids us welcome the new dispensation.

The present volume is written on the lines suggested by the Committee of the Mathematical Association (see *Gazette*, May, 1902), and includes the substance of Euclid, Books I., III., 1 to 34, and IV., 1 to 5, theorems on loci, mensuration, and an appendix on graphs, which, though doubtless a useful introduction to the subject, seems rather out of place here. The remainder of the course is to be published shortly.

The recommendations of the Committee have been very faithfully followed, and the result is in essentials excellent, and should prove extremely useful to teachers. The work has been so well done that we are tempted to find fault with details which might otherwise have passed unnoticed. For we do not regard it as impossible to establish some one standard book on geometry in place of Euclid, and are convinced that in the interests of education such a book is very much to be desired. But while being based on sound lines and wide knowledge, it must be free from superficial blemishes if it is to last for all time, and some of these we are constrained to point out in the present volume.

The definition of a straight line has always been a difficulty. To say that it is "even" is only to substitute one adjective for another. It is better not to define, but to state the chief property, namely, the unique determination by two points. So too the idea of an angle is distinct from, and not less simple than, the idea of amount of turning. It, like straightness, is a fundamental intuitive idea, and should not be defined; but the association of rotation with it is extremely valuable, and it is strange, considering the practical nature of the book, that the suggestion of the Committee has not been followed in this case, and that the opportunity of "illustration by rotation" in dealing with the exterior angles of a polygon is missed. A distinction between different directions of rotation should be made, as without it the reasoning on page 3 is invalid.

It seems an unnecessary piece of conservatism to place the equality of right angles among the axioms, especially as a proof is given on the next page.

A list of abbreviations is given on p. 10, which "may be used" (presumably by students) "in writing out propositions." This may be useful in establishing uniformity, but the continual use of unnecessary abbreviations when not required for compactness is disfiguring to the text and annoying to the reader.

The English language has been called a branch of mathematics, and an English text-book should be free from even time-honoured abuses. The word "shall" as used in enunciations is barbarous and unmeaning, and the authors are not even consistent in its use.

A few minor details may be noted. The name pentagon is used on p. 19 and defined on p. 24. To prove prop. 4 by cutting triangles out of a *double sheet* of paper is rather begging the question. In the construction of a triangle with given sides, it should be pointed out that the circles must *cut*. It is surprising to come suddenly upon the definition of a chord on p. 171 after frequent use has been made of the term and after the definition has been given in its proper place at the beginning of Book III.

The book is well supplied with examples both of a theoretical and of a practical character. R. W. H. T. HUDSON.

Elementary Geometry. J. ELLIOTT, M.A. 4s. (Swan Sonnenschein & Co.)

This is an attempt to supplant Euclid: his form of proof is retained, but his order is disregarded; angles at a point and parallels come first; the triangle, congruent triangles, and an admirable section on the solution of problems; then inequalities, circles, and lastly areas. This order seems excellent in many ways; but, if we are to have much measurement, it will be convenient to take areas much earlier, and I. 47, which comes right at the end, should be taken at the same time as square root in arithmetic. The explanations of the definitions and of the fundamental methods employed, and the hints about proving theorems and solving problems, are all admirable; in some cases the mathematician may think them too obvious to be worth reading, but to the non-mathematician and less experienced teacher they will prove of great assistance; as for the pupil he will not read them—they are too long. In fact, the whole book is too long and too full for actual school use. Such a book ought to be the teacher's companion to a more concise book to be placed in the hands of the class.

An angle is defined as the difference of two directions, and parallel straight lines as lines in the same direction. The author defends these on the grounds that they have proved successful in class teaching. Even considering the clear illustrations that lead up to the definition of an angle, it is impossible to accept it without a shudder. Would it not have been wiser to give the illustrations, and say that an angle cannot be satisfactorily defined but that certain axioms will be assumed?

It is a pity the author has not shown how experimental work should be interwoven with the formal geometry. Many numerical examples and illustrations might have been given.

The size of the book (there are 268 pages, and the ground covered is only Euclid I.-IV.) might have been decreased with advantage by using the usual abbreviations. In general, it is easier to follow a proof in which they are used. The printing and figures are not all that could be desired. There are several misprints (mostly unimportant, it is true), and in one figure several lines are left out. Why is a figure often repeated on the *opposite* page? A. W. SIDONS.

A Philosophical Essay on Probabilities. By PIERRE SIMON, MARQUIS DE LAPLACE (translated by F. W. TRUSCOTT and F. L. EMORY). Pp. iv., 196. 2 dols. 1902. (Wiley, New York; Chapman & Hall.)

Probabilités et Moyennes Géométriques. By E. CZUBER (translated into French by H. SCHUERMANS, with a preface by C. LAGRANGE). Pp. xii., 244. 8 fr. 50. 1902. (Hermann, Paris.)

Wahrscheinlichkeitsrechnung, und ihrer Anwendung auf Fehlerausgleichung, Statistik, und Lebensversicherung. By E. CZUBER. Part I. Pp. 304. 1902. (Teubner, Leipzig.)

"The gambling fraternity will continue to proclaim their belief in luck—and the community on whom they prey will, for the most part, continue to submit to the process of plucking, in full belief that they are on their way to fortune." Thus wrote R. A. Proctor, in the preface to his serious and amusing little volume, *Chance and Luck*. A member of the gambling fraternity who purchases any of the above publications with a view to improving his chances on the turf, at Monte Carlo, or in the Continental lotteries, will be as edified as was the too confiding farmer who bought Ruskin *On Sheepfolds*. An angel from heaven could not convince this class of mind that if a penny has turned up tails ten times running, it is still an even chance that it turns up heads or tails on the next throw. The nearest approach to sanity that such a being might conceivably exhibit would be the suggestion that the coin in all probability was loaded. To him, therefore, a philosophical treatise such as that of Laplace will in nowise appeal. The books under notice are for those "who await no gifts from fate," even if it is too soon to say that, as Arnold goes on to add, "they have conquered fate." Unfortunately, the intrinsic nature of the subject is one that may mislead the acutest minds as well as the ordinary layman. As De Moivre pointed out, the reason of the layman's confidence is the deceptive character of the problems of chance, which seem as if they but require ordinary sound sense for their solution. Even when

error has crept into solutions by expert hands, it is often exceedingly difficult to detect the source of the misconception. This will, no doubt, account in a large measure for the fascination which this subject has exercised on piercing intellects, and perhaps that logical divine, Archbishop Whately, had this in his mind when he sat down to compose his ingenious pamphlet: *Historic Doubts Respecting Napoleon Bonaparte*.

Roughly speaking, Pascal and Fermat divide the glory of founding the theory, the views of the latter being, indeed, more accurate, as far as fundamental principles are concerned, than those of his friend and rival. The famous numbers of James Bernouilli make their first appearance in his *Ars Conjectandi*. Cotes "of Trinity," of whom Newton said that if he had lived we should have learned something, was the first to explore the theory of errors. The theory of recurring series saw the light in De Moivre's *Doctrine of Chances*. Thomas Simpson carried on the work of James Dodson (who wrote a century before) in his *Laws of Chance and Theory of Annuities and Reversions*. All this and more we may find in the pages of Todhunter's classic treatise on the history of the subject. We may add *en passant* that it is surely time that we had a reprint of this famous volume, which has been out of print for many years, and can only be purchased at a price beyond the reach of the shallow purse.

Just as Laplace published a popular *Système du Monde* as a pendant to his *Mécanique Céleste*, so to his great *Théorie Analytique des Probabilités* he writes for his contemporary public the book which heads our list. The translation under notice cannot be said to be entirely successful. On the whole it is faithful and straightforward enough, but it is uneven, and in many places ungraceful. The table of errata would seem to betray the cause. For "Pline" read "Pliny"; for "sun" read "soil"; for "primary" read "prime." The translators are professors respectively of Germanic languages and of Applied Mathematics. It would be interesting to know whether the Professor of German languages confused *soleil* with *sol*, and did not know of the existence of *prime* numbers, and if Pliny was to him but a shadowy personality. The Teuton Ph.D., who knows the value attached to a "Von," should not have deprived De Moivre of his badge of aristocracy, shadowy though it may have been. In fact, his name is generally written *Demoivre*. Roger Cotes is presented with a circumflex—Côtes—though Leicester born and bred. For Leibnitz we have "Liebnitz, always led by a singular and very loose metaphysics" (!) where Leibniz or Lubenicz are the only lawful variations. The niece of Pascal, heroine of the miracle of the Holy Thorn, was surely Marguerite Perier and not Mlle. Perrier. Ticho-Brahé looks very strange to English eyes. The few passages in which mathematics is directly introduced are none the easier to read for being expressed, as in the following instance: We "have Z equal to the binomial $T-1$; the product of V by the n th power of Z will thus be equal to the product of V by the development of the n th power of the binomial $T-1$." Surely the translators might have taken the bull by the horns, and, to mingle metaphors, they might have given us in approximately one page this "linked sweetness long drawn out" into 6½ pages.

The best English summary of Laplace's reasoning in his great book on probabilities is given by De Morgan in the *Encyclopaedia Metropolitana*. The manner of using his results, as De Morgan said elsewhere, may be described to a person who has a nodding acquaintance with decimal fractions. We may say that a knowledge of vulgar fractions will almost suffice for the comprehension of the book under notice. It deals with the general principles underlying the doctrines of probability and hope, and their application to games of chance, to astronomy, testimony, the decisions of assemblies and tribunals, and to tables of mortality. Among the more interesting of the allusions to contemporary topics of discussion is that to Pascal's argument, reproduced under a geometrical form by Craig, "the English mathematician." "Witnesses declare that they have it from Divinity, that in conforming to a certain thing one will enjoy, not one or two, but an infinity of human lives. However feeble the probability of the proofs may be, provided that it be not infinitely small, it is clear that the advantage of those who conform to the prescribed thing is infinite, since it is the product of this probability and an infinite good; one ought not then to hesitate to procure for oneself this advantage." Laplace puts his tongue in his cheek and the happy lives in an urn, and with some unction demolishes the argument of Pascal. The book is full of interest even in its present dress. Its perusal is, of course,

infinitely less arduous than the task of mastering the *Théorie Analytique*. As Morgan Crofton, referring to the greater work, concludes his article in the *Encyclopædia Britannica*: "It is, and will long continue to be, one that must be attempted by all who desire to understand and to apply the theory of probability."

The second of the books with which this notice is headed is a translation of Czuber's *Geometrische Wahrscheinlichkeiten und Mittelwerthe*, published at Prague in 1884—a work on local probability which received the honour of special mention in the article to which we have just alluded. It consists of a series of problems—*joyeux devis*—many of which lead to the most fascinating of paradoxes, due in general to the sense in which the solver understands the much disputed term "at random." It is gratifying to find that the author has largely drawn on the hundreds of problems in the *Educational Times* connected with the names of Blackwood, Clarke, Mc'Coll, Seitz, Sylvester, Watson, Wolstenholme, and Woolhouse. It is also gratifying to find in its right place a name which naturally received but a bare mention in the *Encyclopædia* article, that of Morgan Crofton himself, whose great memoir in the *Phil. Trans.* of 1868 entitled him to be called the father of this branch of investigation. As Herr Czuber justly observes: "Crofton's methods are alike distinguished by elegance and generality; to him is the merit of having been the first to avail himself of these methods in analysis, especially in the integral calculus."

The translation should therefore be well received in this country by those who are interested in the history of this branch of the subject. A full notice of the more general treatise by the same author must be deferred until the second volume is in our hands. The first volume contains a general introduction to the theory, with full solutions of a large number of typical examples, including, of course, those which may be called the "classical" problems. He then deals with the theory of errors, and the methods of least squares arising in connection therewith. Examples from different branches of science exhibit the application of this potent weapon in dealing with the manipulation of observations. Statistics and life assurance appear to be adequately dealt with, but we cannot speak of the treatment of the latter as a whole, as, according to German custom, the first volume ends abruptly in the middle of a paragraph. From what we have seen of Vol. I., the book is clearly of unusual merit. But to return. Interest in questions of "local probability" was first aroused by Buffon's famous "problem of the needle," and kindred problems. Until the English school took up the subject, aided and abetted by a few enthusiasts such as Lemoine, Jordan, and Lalanne, the thoughts of mathematicians were mainly directed to a few solitary and disconnected questions. Even Buffon, whose gifts were not of a mathematical order, cried out for more geometry in dealing with his needle, and his instinct was shared by Herr Czuber, as may be seen from the title of the book. In the days of Buffon it would have been impossible to prophesy the full scope which in the future was to be given to questions of this type. Yet so wide a field is covered by the applications of the theory, that it is of prime importance that we should feel some confidence as to its bases. One of the many difficulties surrounding the initial stages of investigation is very well put by Mr. Venn in his *Logic of Chance*: "It consists in choosing the class to which to refer an event, and therefore judging of the event and the improbability of foretelling it *after it has happened*, and then transferring the impressions we experience to a supposed contemplation of the event beforehand." And in illustration he gives the case of the man who exhibited a bull's eye on a small target as the result of his skill with an old fowling-piece at a distance of a hundred yards. The braggart had suppressed a fact which, after all, had something to do with the case. He had fired at the door, and had *then* placed a small chalk circle around the aperture made by the shot.

Again, let us take Chrystal's definition of probability. "If on taking a very large number N out of a series of cases in which an event A is in question, A happens on pN occasions, the probability of the event A is said to be p ." Apply it to the question: I throw two dice, what is the chance that I throw at least one six? There are eleven unfavourable to thirty-six favourable cases, giving the probability $\frac{11}{36}$. But as far as the definition is concerned, why should we not treat it as follows: The number of combinations of the numbers on the dice is twenty-one, of which six are favourable, so that the probability is $\frac{6}{21}$?

Our definition is incomplete unless we add "assuming that the possible cases are equally probable." Which brings us to a *definition of the probable by the probable*. Or, again, the event may or may not happen—two possible cases, one favourable, hence the probability is $\frac{1}{2}$. Let us take another illustration from Bertrand. What is the chance that a random chord in a circle may be greater than the side of the inscribed equilateral triangle? If one extremity of the chord be known, the probability is unaffected thereby. The two sides of the equilateral triangle through this point make with the tangent at the point three angles of 60° . To be greater than the side of the triangle the chord must lie within the angle of the triangle. The chance of this is $\frac{1}{3}$. On the other hand, if we are given the direction of the chord, the probability is thereby unaffected. It is easily seen that the chance of this again is $\frac{1}{3}$. But we may also argue: to choose a random chord is to choose at random its middle point. The mid point of the chord must then be at a distance from the centre less than half the radius, i.e. within a circle one-fourth of the area of the original circle. The number of points within a circle four times less is four times less. Hence the chance is $\frac{1}{4}$.

Fortunately it is possible to find a portion of firm ground on which to take a stand in the midst of this quaking morass of haunting doubt. Were it not so, as M. Poincaré points out (*La Science et l'Hypothèse*, p. 217), we should involve in one common ruin the countless problems of science, in which probabilities play so important a part. It is not enough to say, "ce que je sais, ce n'est pas que telle chose est vraie, mais que le mieux pour moi est encore d'agir comme si elle était vraie. . ." We cannot, and we must not, condemn *en bloc*; discussion is as obligatory as it is inevitable. And in this connection it would be well for us to ponder once more over Kant's reflexions in the 4th Section of the *Antinomy of Pure Reason*. I am glad to be able to refer the reader to the extraordinarily interesting work of M. Poincaré. His concluding words are well worth bearing in mind: "There are certain points which seem well established. In any question of probability we must start with an hypothesis or a convention which will always contain something arbitrary. In the choice of this convention we can only be guided by the principle of sufficient reason. Unfortunately, this principle is entirely vague and elastic, and . . . assumes many different forms. The form under which we meet it most frequently is the belief in continuity, a belief which it would be difficult to justify by apodictic reasoning, but without which all science would be impossible. The problems to which the theory of probabilities can be profitably applied are those in which the result is independent of the initial hypothesis, with the sole proviso that the hypothesis satisfies the condition of continuity." The reader who is still dissatisfied must remember that, after all, the very name of the calculus of probabilities is in itself a paradox of the purest water. "La probabilité, opposée à la certitude, c'est ce qu'on ne sait pas, et comment peut-on calculer ce que l'on ne connaît pas?"

Histoire des Mathématiques dans l'Antiquité et le Moyen Age. By H. G. ZEUTHEN (translated from the Danish and German by J. MASCART). Pp. xvi., 296. 1902. (Gauthier-Villars.)

This translation of Professor Zeuthen's learned and popular work has the advantage of containing many additional notes and references from the pen of M. Tannery. There are various alterations in the sections on Trigonometry, due to the author's appreciation of Braunnühl's *Vorles. über Geschichte der Trigonometrie*. Speaking roughly, the volume before us deals mainly with Greek geometry. Euclid's elements from the *point de repère* from which are examined the logical forms so closely followed by the Greeks. The intrinsic nature of these forms apart from their significance to the minds of the early geometers. The student of Euclidean geometry will find the comments of the author of signal interest at the present time. Professor Zeuthen devotes some fifty pages to the investigations of the Arabs and Hindoos. The special aptitude of the Hindoos for numerical calculation is clearly expounded, as is also their extraordinary power of assimilating what they received from the West and applying the results to the production of original and independent results. This power the Arabs did not possess. None the less do we owe it to them that the cloisters of the Middle Ages by means of Arabian manuscripts were able by their translations into Latin to keep the lamp of science burning amid the darkness which was the natural

accompaniment of a general paralysis of intellectual activity. We hope that M. Mascart will be able to induce M. Gauthier-Villars to publish Professor Zeuthen's earlier work on the theory of Conics in antiquity.

Comptes rendus du deuxième congrès international des mathématiciens. (Paris, Aug. 6-12, 1900.) Edited by E. Duporcq. Pp. 450. 16fr. 1902. (Gauthier-Villars.)

This volume consists of the proceedings at the last Congress over which the illustrious Hermite presided. Space forbids our drawing attention to more than a few of the lectures and communications made to the Congress. Of the five formal lectures, one is by M. Mittag-Leffler on the correspondence between Weierstrass and Sophie Kowalevski. Incidentally the distinguished Swedish mathematician narrates the first words addressed to him on presenting himself as a student to attend Hermite's course at Paris: "Vous avez fait erreur, Monsieur; vous auriez dû suivre les cours de Weierstrass à Berlin. C'est notre maître à tons." This observation, three short years after *la funeste année*, 1870, shows how far the instincts of the Savant were wrought to finer issues than the purely local and patriotic sentiment of the day. M. Hilbert's lecture on the problems which are at present awaiting their solution by the hands of competent mathematicians has been translated into English by Dr. Mary Newson for the Bull. Math. Soc. (July, 1902). The problems are as follows: Cantor's problem of the cardinal number of the continuum; the compatibility of the axioms of arithmetic; the equality of the volumes of two tetrahedra of equal bases and equal altitudes; the problem of the straight line as the shortest distance between two points; Lie's concept of a continuous group of transformations without assuming that the functions defining the group are capable of differentiation; the treatment of the axioms of physics as we treat the axioms of mathematics, placing in the first rank probabilities and mechanics; the irrationality and transcendence of certain numbers; that the zero points of $\sum_{n=0}^{\infty} \frac{1}{n^s}$ have all the real part $\frac{1}{2}$, except the negative integral real zeroes; Riemann's prime number formula; Goldback's theorem, that every integer is expressible as the sum of two primes; is there an infinite number of pairs of primes differing by 2? is $ax+by+c=0$ soluble in prime numbers x and y , where a, b are integral and a prime to b ? to apply the results obtained for the distribution of rational prime numbers to the theory of the distribution of ideal primes in a given number field k ; and so on. M. Poincaré discoursed on the rôle of intuition and logic in mathematics, showing how, while intuition is often the source of discovery, it is logic which harmonises and consolidates the creations of intuition. This address is also published in a separate form (1fr.). Mr. H. Hancock's paper on Kronecker's Modular Systems defines congruences between algebraical integral numbers which are generalisations of the congruences of our elementary theory of numbers. That veteran American, Mr. Artemus Martin, gives new series for the calculation of logarithms, some of which converge with quite unusual rapidity. He also deals with expressions of the form $x^4+y^4+z^4+\dots+u^4=k^4$. The Italians are following the lead of Professor Peano in their discussions on mathematical logic. Signor A. Padua contributes two papers, one on a new irreducible system of postulates for algebra—seven in number, based on the (undefined) integer, its consecutive, and its image (symétrique)—and the other on a new system of definitions for Euclidean geometry—two in number, point and "may be superposed." Signor Veronese pleads for the reduction to a minimum of the postulates in teaching geometry. Professor A. Macfarlane applies space analysis to curvilinear co-ordinates; and Professor Stringham discusses orthogonal transformations in elliptic space. Signor Galdeano advocates the addition to the branches of Mathematics as taught at the Universities the principles of "Mathematical Criticism." It would consist of the study of the historical developments and the ties of kinship which link together the historical and the logical genesis of our knowledge. In a synthetic study of different branches "l'enchaînement des idées" must be fruitful of result. Signor A. Gallardo writes on the application of Mathematics to a complex subject, such as Biology. We must not forget to mention Mr. Fujisawa's engaging article on the Mathematics of the Old Japanese School. Few readers will be unable to discover in this *Compte Rendu* some paper or article in which they will be interested. It is a delightful collection.

Differential- und Integralrechnung. Vol. I. Differentialrechnung. By W. F. Meyer. Pp. xviii., 395. 9m. (Göschen, Leipzig.) Vol. X. Sammlung Schubert.

This interesting volume is well worthy the careful perusal of teachers of the elements of the Calculus. It is especially remarkable for the prominence given at an early stage to the theory of errors. It is not until after some 70 pages of introductory matter that the author approaches the differential coefficient, the first chapter being devoted entirely to the fundamental conceptions. He begins with a theorem on the limits of a^n and a^m for $n = \alpha$, applying them to the summation of a G.P. He proceeds at once to use these results in determining the areas of plane curves and the volumes of solids of revolution. In the same way other limits, such as $\frac{\sin \theta}{\theta}$, $\frac{\sin \theta}{\Pi \cos \frac{\theta}{2^n}}$, lead us to the tangents to conics. A section on

the Binomial Theorem is followed by the determination of the equation of the tangents to a parabola of the m^{th} order and the elementary notions of integral functions. The second part is quite a monograph in itself on the development of series. Rolle's theorem and its applications, Taylor's and Maclaurin's Theorem, are treated at length; and the final sections form an adequate discussion on the convergence and divergence of series and of infinite products. The second volume will contain the applications of the differential calculus to curves and surfaces, and also a series of historical notes. To it we look forward with considerable interest. We can heartily commend this attractive little volume to the attention of those who are not satisfied with the ordinary introductions to elementary analysis.

PROBLEMS.

460. [R. 2. b. γ.] Find the centre of gravity of the part cut from a solid sphere by two diametral planes inclined at angle 2α to each other. ANON.

461. [M¹. 3. b.] MM' is a chord of a circle, centre O , parallel to a fixed diameter AB ; MP is perpendicular to OM' . Find the areas of the locus of P and of the envelope of the line MP . What do they become when we replace the circle and its diameter by an ellipse and its major axis?

E. N. BARISIEN.

462. [K. 12. b. α.] Describe three circles, mutually tangent, to pass through three given points and touch a circle including the three points. A. B.

463. [L¹. 17. α.] Shew that the equation to a circumconic of the triangle ABC can be written in the form

$$\frac{a}{pa} + \frac{b}{q\beta} + \frac{c}{r\gamma} = 0,$$

where p, q, r are the lengths of the focal chords parallel to BC, CA, AB .

J. J. MILNE.

464. [K. 10. α.] A, B are two fixed points, P a variable point, on a circle; find the locus of the intersections of BP with the bisectors of the angle BAP .

V. RETALI.

465. [L¹. 6. c.] CP, CD are conjugate radii of an ellipse; $PU, D'V$ the chords of intersection of the ellipse with the circles of curvature at P, D ; shew that CU, CV are conjugate.

C. F. SANDBERG.

466. [K. 20. α.] In a triangle which has $\sum \cot A < 2$, shew that the least angle $> \cot^{-1} \frac{1}{2}$ and the greatest $< 90^\circ$. If $\sum \cot A > 2$, what is the greatest value of the least angle and the least value of the greatest? C. E. YOUNGMAN.

467. [L¹. 3. d.] Two tangents to a rectangular hyperbola meet on a fixed parabola having one asymptote of the hyperbola for axis and the other for the tangent at its vertex. Find the envelope of their chord of contact.

Durham, 1902.

468. [R. 4. c.] A parallelogram $ABCD$, formed of uniform rods of total weight W , smoothly jointed, is held in a vertical plane with AB vertical (A uppermost), and a light elastic string of natural length d and modulus W connects the joints A, C . Find when in equilibrium the length of the string.
Melbourne, 1901.

SOLUTIONS.

400. [K. 2. d; 5. d.] A variable straight line meets two fixed straight lines Ox, Oy in A and B , so that the sum or difference of OA, OB is constant: find the envelope of the Euler circle of the triangle OAB , and the locus of its Lemoine point and the three associated points.

Prove also that the circumcircle of OAB passes through another fixed point besides O , and that its Euler line also passes through a fixed point.

E. N. BARISIEN.

Solution by C. E. YOUNGMAN.

Let ω be the angle AOB . Draw the circle OAB (centre C) and the diameter DD' bisecting AB at M , so that OD is the internal bisector of the angle AOB . Then $DA = DB$, and angle $DAO = \pi - DBO$; therefore, if DA, DB be turned in the same direction through any angle θ to positions Dx, Dy , we have $Ax = By$ and $Ox + Oy = OA + OB$. Thus D is a fixed point when $OA + OB$ is constant; and similarly, D' is fixed when $OA - OB$ is constant. Suppose the former: then, DA', DB' being perpendicular to OA, OB , M lies on $A'B'$ (the pedal line of D for OAB). Let fall AHE, BHF perpendicular to OB, OA ; the circle MEF is the "Euler" circle of OAB (and surely "nine-point circle" is a better name, better known). Now M is the centre of the circle $ABEF$; therefore $ME = MF$, and angle $EMF = 2EAF = \pi - 2\omega$. Hence the triangle MEF has constant angles, and is inscribed to a fixed triangle $OA'B'$; therefore the circle MEF touches twice a fixed conic inscribed in $OA'B'$ (Neuberg, *Proc. Lond. Math. Soc.* xvi. 185). Also, since MEF is, in one position, the orthocentric triangle of $OA'B'$, the foci of the conic are the ortho- and circumcentres of $OA'B'$.

The Euler line CH divides OD in the ratio

$$OH : CD = 2CM : CA = 2 \cos \omega : 1 = \text{const.};$$

i.e. it cuts OD at a fixed point P .

Let $KK_1K_2K_3$ be the Lemoine point and its associates. K_1 is the pole of AB for the circle OAB ; therefore $CM = CA \cos \omega = CK_1 \cos^2 \omega$, and $CM : CK_1 = \text{const.}$ Now C and M move on lines perpendicular to OD ; therefore K_1 on another. K_2 is the pole of a fixed line OB for a set of circles OD , that is, of conics passing through four fixed points— O, D , and the two at infinity; therefore K_2 describes a conic—a hyperbola, for K_2 goes to infinity when C is on OB or at infinity. K_3 , of course, describes an equal hyperbola. As for K , it is the mean centre of O, A, B for multiples AB^2, BO^2, OA^2 , or $a^2 + b^2 - 2ab \cos \omega, b^2, a^2$. Hence its coordinates are given by

$$2x(a^2 + b^2 - ab \cos \omega) = ab^2, \text{ and } 2y(a^2 + b^2 - ab \cos \omega) = a^2b.$$

From these and $a + b = c$ eliminate a and b ; the result is

$$2(x+y)(x^2 + y^2 - xy \cos \omega) = cxy;$$

so that the locus of K is a cubic which has a node at O , and an asymptote parallel to OD' .

413. [K.] Along three straight lines meeting in O at angles α, β, γ , lengths x, y, z are measured: prove that the three points so obtained lie on a straight line if $\Sigma x^{-1} \sin \alpha = 0$, and on a circle through O if $\Sigma x \sin \alpha = 0$.

R. W. H. T. HUDSON.

Solution by C. V. DARELL, E. FENWICK, and others.

Let P, Q, R be the three points.

Then since P, Q, R are collinear, we have $\Sigma \triangle QOR = 0$;

$$\therefore \Sigma yz \sin \alpha = 0.$$

$$\therefore \Sigma x^{-1} \sin \alpha = 0.$$

If P, Q, R lie on a circle through O , invert with respect to O, K being the constant of inversion, then the points whose distances from O measured along the three fixed lines are $\frac{K^2}{x}, \frac{K^2}{y}, \frac{K^2}{z}$ must be collinear.

Therefore, using previous result,

$$\Sigma x \sin \alpha = 0.$$

BOOKS, ETC., RECEIVED.

Changes in the Regulations for Responsions. Oxford. (From H. T. GERRANS, Chairman of the Board.)

Schedules for Geometry. Cambridge Local Examinations. (From DR. MACALISTER and DR. KEYNES.)

The Teaching of Geometry. By W. J. DOBBS. Pp. 10. (Reprint from *Journal of Education*), Nov. 1902.

Aufgabensammlung. By E. BARDEY. Pp. viii., 395. 3 m. 20. (Teubner.)

Leçons de Mécanique Élémentaire. By P. APPELL ET J. CHAPPUIS. Pp. viii., 176. 2 fr. 1903.

Annuaire pour l'an 1903 (Bureau des Longitudes). Pp. viii., 668, 120. 1 fr. 50. (Gauthier-Villars.)

Bulletin of the American Mathematical Society. Edited by F. MORLEY and others.

Nov. 1902. *Second Report on Recent Progress in the Theory of Groups.* E. A. MILLER.

Dec. 1902. *Commutator Sub-Groups of Groups whose orders are powers of primes.* W. B. FITE. *Irregular Determinants.* L. L. HEWES. *Projections of the Absolute Acceleration in Relative Motion.* G. O. JAMES. *Infinitesimal Deformation of the Skew Helicoid.* L. P. EISENHART. *Integrability by Quadratures.* S. EPSTEIN.

Il Pitagora. Edited by G. FAZZARI. Oct.-Nov., 1902.

Periodico di Matematica. By G. LAZZERI. Nov.-Dec., 1902.

Supplement al Per. di Mat. ... Nov. 1902.

Gazeta Matematica. Edited by ION IONESCU. Nov., 1902.

The American Mathematical Monthly. Edited by B. F. FINKEL and L. E. DICKSON. Nov., 1902.

A Text-Book of Field Astronomy for Engineers. By G. C. COMSTOCK. Pp. x., 202. 1902. (Wiley, New York.)

Mathematical Correspondence:—Robert Simson, Matthew Stewart, James Stirling. (Proceedings Ed. Math. Soc., Vol. xxi.) By J. S. MACKAY.

Sur une Génération du Limacon de Pascal. By E. N. BARIEN. Pp. 124-151. Association Française pour l'avancement des Sciences. 1901.

A Mathematical Solution Book. By B. F. FINKEL. 4th edition. Pp. xvi., 550. 1902. (Kibler, Springfield, Mo.)

ERRATA.

p. 224, for Note 109 read 117.

p. 225, for Note 110 read 118.

p. 225, to Note 110 attach at end—E. L. BULMER.

PROCEEDINGS OF THE EDINBURGH MATHEMATICAL SOCIETY.

Vol. XX., pp. 4, 77. Session 1901-1902. Williams and Norgate.

R. E. ALLARDICE.

On some systems of conics connected with the triangle.

H. F. BLICHFELDT.

Demonstrations of a pair of Theorems in Geometry.

A. G. BURGESS.

Theorems in connection with lines drawn through a pair of points parallel and antiparallel to sides of a triangle.

J. W. BUTTERS.

Notes on Decimal Coinage and Approximation.

H. S. CARSLAW.

On use of Fourier's Series in the Problem of the Transverse Vibrations of Strings.

On the Theorem that $mx^{m-1}(x-1) > x^m - 1 > m(x-1)$ unless $0 < m < 1$, and then it is $< x^m - 1 < m(x-1)$.

L. CRAWFORD.

A Proof of Rodrigues' Theorem,

$$\sin n = \frac{n}{1 \cdot 3 \cdot 5 \dots 2n-1} \left(\operatorname{cosec} x \frac{d}{dx} \right)^{n-1} \sin^{2n-1} x,$$

etc.

On a property of certain Circulating Decimals.

E. A. GIBSON.

The Second Integral of Mean Value.

J. S. MACKAY.

History of a Theorem in Elementary Geometry.

Note on the Theorems of Menelaus and Ceva.

T. MUIR.

The Law of Extensible Minors and certain Determinants.

R. F. MUIRHEAD.

Note on the theory of the rolling of one rigid surface on another.

Note on the Theorems of Menelaus and Ceva.

Constructions connected with Euc. vi. 3 and A and the $\odot$ of Apollonius.

Geometry of the Isosceles Trapezium and the Contraparallelogram, with applications.

C. TWEEDIE.

Anallagmatic Curves, I.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."—BACON.

President :

J. FLETCHER MOULTON, K.C., M.P., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.

R. B. HAYWARD, M.A., F.R.S.

Prof. W. H. H. HUDSON, M.A.

R. LEVETT, M.A.

Professor A. LODGE, M.A.

Professor G. M. MINCHIN, F.R.S.

Treasurer :

F. W. HILL, M.A., City of London School, London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.

H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

R. F. DAVIS, M.A.

J. M. DYER, M.A.

C. GODFREY, M.A.

G. HEPPLE, M.A.

F. S. MACAULAY, M.A., D.Sc.

C. E. M'VICKER, M.A.

J. C. PALMER.

Rev. J. J. MILNE, M.A.

A. E. WESTERN, M.A.

C. E. WILLIAMS, M.A.

THE MATHEMATICAL ASSOCIATION, formerly known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all masters and mistresses, who are interested in promoting good methods of mathematical teaching. Such an Association should become a recognized authority in its own department, and should exert an important influence on methods of examination.

General Meetings of the Association are held in London once a term, and in other places if desired. At these Meetings papers on elementary mathematics are read, and any member is at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

"The *Mathematical Gazette*" is the organ of the Association. It contains—

- (1) ARTICLES, on subjects within the scope of elementary mathematics.
- (2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books.
- (3) REVIEWS, at present the most striking feature of the *Gazette*, and written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell rather on the general development of the subject, than upon the part played therein by the book under notice.
- (4) PROBLEMS AND SOLUTIONS, generally selected to show the trend of investigation at the universities, so far as is shown in the most recent scholarship papers. Questions of special interest or novelty also find a place in this section.
- (5) SHORT NOTICES, of books not specially dealt with in the REVIEWS.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "The *Mathematical Gazette*."

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 38.

MARCH, 1903.

1s. 6d. Net.

The Mathematical Association.

NOTICES of papers or discussions for the General Meetings in May and October should reach the Hon. Secretaries not later than April 1st and September 1st respectively.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, 21 Argyll Place, Aberdeen.

All other correspondence concerning the Gazette should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

Correspondence relative to the Mathematical Association should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, St. Paul's School, London, W.; H. D. ELLIS, 12 Gloucester Terrace, Hyde Park, W.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied, except on payment of 1s. per copy

Obituary.

HON. MEMBERS.

The Rev. N. M. FERRERS, D.D., F.R.S., Master of Gonville
and Caius College, Cambridge.

The Rev. H. W. WATSON, M.A., F.R.S., Berkeswell Rectory,
Coventry.

MEMBERS.

The Most Reverend the LORD ARCHBISHOP OF CANTERBURY,
D.D.

R. B. HAYWARD, M.A., F.R.S., Ashcombe, Shanklin, Isle
of Wight.

CONTENTS.

	PAGE
THE REPORT OF THE COMMITTEE, AND INCOMMENSURABLES. Prof. M.	
J. M. HILL - - - - -	253
ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION, - - -	259
MATHEMATICAL NOTES. Prof. G. H. BRYAN; H. L. TRACHTENBERG; C. S.	
JACKSON; R. F. DAVIS, - - - - -	260
REVIEWS. Prof. G. B. MATHEWS; Prof. G. H. BRYAN; Prof. E. B.	
ELLIOTT; Prof. C. J. JOLY, - - - - -	263
PROBLEMS, - - - - -	269
SOLUTIONS, - - - - -	270
BOOKS, ETC., RECEIVED, - - - - -	276
ERRATA, - - - - -	276

Forthcoming Articles and Reviews by:—

Prof. A. C. DIXON; Prof. T. J. P^a BROMWICH; Prof. A. LODGE; Prof.
G. B. MATHEWS; W. G. BELL, M.A.; J. H. GRACE, M.A.; R. W. H. T.
HUDSON, M.A.; E. M. LANGLEY, M.A.; Rev. W. H. LAVERTY, M.A.;
R. M. MILNES, M.A.; W. N. ROSEVEARE, M.A.; E. T. WHITTAKER,
M.A.; W. P. WORKMAN, M.A.; and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. *No. 8 is out of print.*
A few numbers may be obtained from the *Editor* at the following prices,
post free:

No. 7, 5s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 15s.

These will soon be out of print and very scarce.

A column will be opened in the next number of the 'Gazette' for
sale or exchange of Mathematical works and periodicals.

THE MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

MARCH, 1903.

No. 38.

THE REPORT OF THE COMMITTEE, AND INCOMMENSURABLES.

THE Report of the Committee of the Mathematical Association on the Teaching of Geometry having been published, I desire to draw attention to their treatment of the subject of ratio.

The Committee think (§ 48 of the Report) "that an ordinary school course should not be required to include incommensurables"; but they do not provide any place for such a treatment of ratio as will form a preparation for that subject. I fear the result of this will be (though probably that was not intended) that the whole subject will be entirely ignored, not only in the case of those who are following an ordinary school course, but also in the case of those who wish to obtain accurate notions of the Infinitesimal Calculus; and of those who are preparing to take Honours in Mathematics at the Universities.

Let me begin by endeavouring to clear away a misconception. The subject of ratio is an algebraic one, not a geometric one. Its treatment in the Fifth Book of Euclid has given rise to the idea that there is a geometric treatment of the subject as distinct from an algebraic one; and it would seem from the use of the word "algebraic" in § 47 (2) of the Committee's Report that they adopt this view.

Euclid employs a segment of a straight line to denote a magnitude, and more recently figures have been employed to illustrate what De Morgan has called the relative multiple scale of two magnitudes of the same kind. But these do not form an essential part of the reasoning. They are merely aids to the beginner in following the reasoning. All that Euclid assumes is that if A and B are two magnitudes of the same kind, and if r and s are any two integers, then it is possible to tell whether rA is greater than, equal to, or less than sB . This assumption seems to me to take the place of a definition of what is meant when we say that two magnitudes are of the same kind. However this may be, from this point onwards the whole of Euclid's work in the Fifth Book is strictly algebraic.

Leaving this point, it seems to me that the Report of the Committee would have given more help to teachers if they had sketched out an adequate "theory of measurement of lengths of lines and areas of rectangles for cases in which the lines and the sides of the rectangles are commensurable," § 47 (1), and if they had actually given the "algebraical treatment of of ratio and proportion for commensurables," § 47 (2), in a form which they

regard as sufficient. The teacher would then see how he was expected to explain the propositions

$$pA : qA = p : q \quad [\text{Euc. X. 5}],$$

and

$$rA : rB = A : B \quad [\text{Euc. V. 15}],$$

where A and B are magnitudes and p, q, r are integers. There is no difficulty about either of these propositions, but in the case of each of them there exists the danger that the beginner may consider himself entitled to treat the ratios as quotients of numbers containing a common factor, which may be removed.

In the elements of a subject it is necessary before all things that essential distinctions should be observed. There is nothing wanting in definiteness in Euclid's treatment of these subjects when expressed in a modern form, but it is very difficult to induce any one now to look at Euclid's Fifth Book, even though the cause for its difficulty has been discovered and removed. I doubt whether there exists any other treatment equally simple and equally clear. But the Committee have definitely rejected Euclid's line of argument, and it is therefore advisable to examine what they recommend.

I select (as an example of the results to which their line of treatment leads) § 55 of their Report, in which the Committee give their proof of the proposition: "If two triangles (parallelograms) have one angle of the one equal to one angle of the other, their areas are proportional to the areas of the rectangles contained by the sides about the equal angles."

The Committee say,

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC \cdot AG}{\frac{1}{2}EF \cdot DH} = \frac{BC \cdot AG}{EF \cdot DH}$$

and then, since

$$\frac{AG}{DH} = \frac{AB}{DE},$$

they say

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC \cdot AB}{EF \cdot DE}.$$

I call attention to the omission of some intervening steps in the argument. It should be shown that

$$\frac{\text{rect. } BC \cdot AG}{\text{rect. } EF \cdot DH}$$

can be expressed as the product of the ratios $\frac{BC}{EF}$ and $\frac{AG}{DH}$; and then that

the product of the ratios $\frac{BC}{EF}$ and $\frac{AB}{DE}$ can be expressed as

$$\frac{\text{rect. } BC \cdot AB}{\text{rect. } EF \cdot DE}.$$

How does the Committee justify the passage from the ratio of the areas of the rectangles to the product of the ratios of the sides, and conversely? I presume that it is implied that BC, AG, EF, DH are to be treated as commensurables and then the usual rules for the multiplication of rational fractions are to be applied. I cannot help feeling that the effect on the mind of the beginner will be that, after having had demonstrated to him certain rules for dealing with rational fractions, he has been told to apply these to other magnitudes which are not rational fractions. An undeveloped mind will probably accept this without demur, but its effect must be confusing. Possibly "the algebraic treatment of ratio and proportion for commensurables," which the Committee refer to, may be of such a nature that my objections may be met. But in the meanwhile I desire to suggest an alternative proof of the above proposition, and to indicate the way in which I think the whole of this part of the subject should be treated.

1. The fundamental problem in this part of the subject is to determine two straight lines having the same ratio as two equiangular parallelograms. Place the parallelograms so as to have a common angle, and the sides at the common angle in the same directions, then we have Fig. 1.

The two parallelograms are $ABCD$, $AEFG$. Produce DC to cut EF at H . Let AH cut CB at K . Through K draw a parallel to AE to cut AG at L , and EF at M . Then

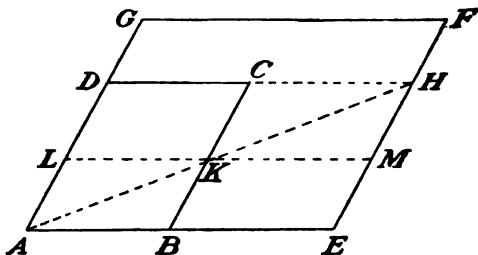


FIG. 1.

$$ABCD = ABKL + LKCD = ABKL + BKME = AEML;$$

We have now constructed on AE , a side of the parallelogram $AEFG$, another parallelogram $AEML$ equal to $ABCD$ and equiangular with $AEFG$.

$$\therefore \frac{AEFG}{ABCD} = \frac{AEFG}{AEML} = \frac{AG}{AL}.$$

Now AG is a side of one of the parallelograms, and is therefore given. The line AL is found thus:

$$\frac{AE}{AB} = \frac{AH}{AK} = \frac{AD}{AL};$$

$$\therefore \frac{AE}{AB} = \frac{AD}{AL}.$$

Hence AL is the fourth proportional to the sides AE , AB , AD of the given parallelograms.

The length of AL depends only on the lengths of certain of the sides of the given parallelograms. *It does not depend at all on the value of the common angle.* If therefore we make that common angle a right angle and draw the rectangle whose sides are equal to AB , AD , and the rectangle whose sides are equal to AE , AG , we prove as above that

$$\frac{\text{rect. } AE \cdot AG}{\text{rect. } AB \cdot AD} = \frac{AG}{AL};$$

$$\therefore \frac{\text{parallelogram } AEFG}{\text{parallelogram } ABCD} = \frac{\text{rect. } AE \cdot AG}{\text{rect. } AB \cdot AD}.$$

2. The line AL can also be identified thus:

$$\therefore \frac{AE}{AB} = \frac{AD}{AL};$$

$$\therefore \text{rect. } AB \cdot AD = \text{rect. } AE \cdot AL.$$

Now the sides of the parallelogram $AEFG$ about A are AG , AE , whilst the sides of the parallelogram $ABCD$ about A are AB , AD . If we select one of the sides of the parallelogram $AEFG$, say AE , and on it describe a rectangle equal to the rectangle $AB \cdot AD$, then the other side of it is the line AL ; and the ratio of the parallelogram $AEFG$ to the parallelogram $ABCD$ is the same as that of the side of the parallelogram $AEFG$, which was not selected, viz. AG to the line AL which was found.

3. From Art. 1 it can be deduced that if two triangles have an angle equal, then their areas are proportional to the rectangles contained by the sides including the equal angles. That the areas of two triangles have the same ratio as the rectangles contained by their bases and altitudes, follows from the fact that the triangles are the halves of the rectangles.

4. The next step is to find two straight lines which are in the same ratio as the areas of two squares. This is included in Art. 1, but the result can be stated in a convenient form. Here we suppose $\hat{A}$ to be a right angle, $AE=AG$ and $AB=AD$. Then the equation of Art. 1

$$\frac{AE}{AB} = \frac{AD}{AL}$$

becomes

$$\frac{AE}{AB} = \frac{AB}{AL};$$

i.e. if AL be a third proportional to AE, AB , then

$$\frac{\text{the square on } AE}{\text{the square on } AB} = \frac{AE}{AL}$$

5. The next step is to prove that the areas of similar triangles are proportional to the squares on corresponding sides.

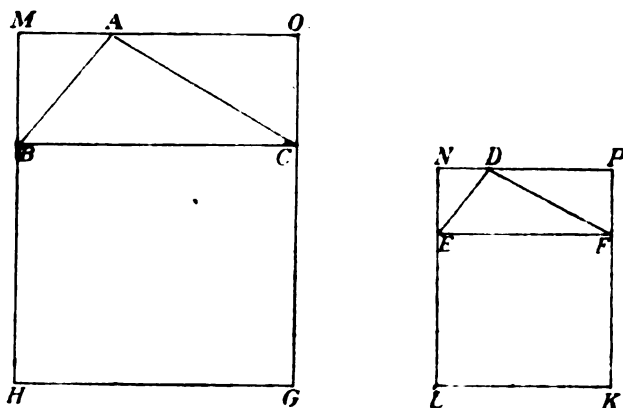


FIG. 2.

Let ABC, DEF (Fig. 2) be similar triangles, so that

$$\hat{BAC} = \hat{EDF}, \quad \hat{CBA} = \hat{FED}, \quad \hat{ACB} = \hat{DFE},$$

and

$$BC : EF = CA : FD = AB : DE.$$

On BC, EF describe the squares $BCGH, EFKL$. Also on BC describe the rectangle $BCOM$ which has a side passing through A , and on EF the rectangle $EFPN$ which has a side passing through D . We have to prove

$$\frac{\triangle ABC}{\triangle DEF} = \frac{\text{square on } BC}{\text{square on } EF}$$

But since

$$BCOM = 2\triangle ABC,$$

$$EFPN = 2\triangle DEF;$$

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BCOM}{EFPN}.$$

We have therefore to prove

$$\frac{BCOM}{EFPN} = \frac{BCGH}{EFKL},$$

$$\text{i.e. } \frac{BCOM}{BCGH} = \frac{EFPN}{EFKL}.$$

but
$$\frac{BCOM}{BCGH} = \frac{MB}{BH} = \frac{MB}{BC},$$

and
$$\frac{EFPN}{EFKL} = \frac{NE}{EL} = \frac{NE}{EF}.$$

Hence it is necessary to prove

$$\frac{MB}{BC} = \frac{NE}{EF},$$

but
$$\frac{BC}{BA} = \frac{EF}{ED}.$$

Hence it is necessary to prove

$$\frac{MB}{BA} = \frac{NE}{ED},$$

but the triangles MBA and NED have

$$\begin{aligned} \hat{BMA} &= \text{rt. } \angle = \hat{END}, \\ \hat{MBA} &= \text{rt. } \angle - \hat{ABC} = \text{rt. } \angle - \hat{DEF} = \hat{NED}; \\ \therefore \hat{BAM} &= \hat{EDN}. \end{aligned}$$

Therefore the triangles MBA and NED are similar ;

$$\begin{aligned} \therefore \frac{MB}{BA} &= \frac{NE}{ED}, \\ \therefore \frac{\triangle ABC}{\triangle DEF} &= \frac{\text{square on } BC}{\text{square on } EF}. \end{aligned}$$

Hence the proposition is demonstrated.

6. The next step is to show that if A, B, C, D are 4 straight lines in proportion, then squares on A, B, C, D are also in proportion.

Find two other lines X and Y , such that

$$\frac{A}{B} = \frac{B}{X} \text{ and } \frac{C}{D} = \frac{D}{Y}$$

Then since $\frac{A}{B} = \frac{C}{D}$ $\therefore \frac{B}{X} = \frac{D}{Y}$; $\therefore \frac{A}{X} = \frac{C}{Y}$

But
$$\frac{\text{square on } A}{\text{square on } B} = \frac{A}{X} \quad \frac{\text{square on } C}{\text{square on } D} = \frac{C}{Y}, \text{ by Art. 4.}$$

$$\therefore \frac{\text{square on } A}{\text{square on } B} = \frac{\text{square on } C}{\text{square on } D}$$

Conversely, if A, B, C, D be four straight lines, such that the squares on A, B, C, D are in proportion, then will A, B, C, D be in proportion.

Find a straight line E such that $\frac{A}{B} = \frac{C}{E}$;

$$\therefore \frac{\text{square on } A}{\text{square on } B} = \frac{\text{square on } C}{\text{square on } E};$$

but
$$\frac{\text{square on } A}{\text{square on } B} = \frac{\text{square on } C}{\text{square on } D};$$

$$\therefore \frac{\text{square on } C}{\text{square on } D} = \frac{\text{square on } C}{\text{square on } E};$$

$$\therefore \text{square on } D = \text{square on } E.$$

From this it follows that $D=E$;

$$\therefore \frac{A}{B} = \frac{C}{D}.$$

Hence A, B, C, D are in proportion.

7. It remains only to prove that the areas of similar rectilineal figures are proportional to the squares on corresponding sides.

Let $A_1B_1C_1D_1, A_2B_2C_2D_2$ be similar figures, so that $\hat{A}_1 = \hat{A}_2, \hat{B}_1 = \hat{B}_2, \hat{C}_1 = \hat{C}_2, \hat{D}_1 = \hat{D}_2$; and $A_1B_1 : A_2B_2 = B_1C_1 : B_2C_2 = C_1D_1 : C_2D_2 = D_1A_1 : D_2A_2$.

To prove

$$\frac{\text{area } A_1B_1C_1D_1}{\text{area } A_2B_2C_2D_2} = \frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}.$$

Let the figures be divided into the same number of similar triangles, as in Fig. 3 (Euc. VI. 20).

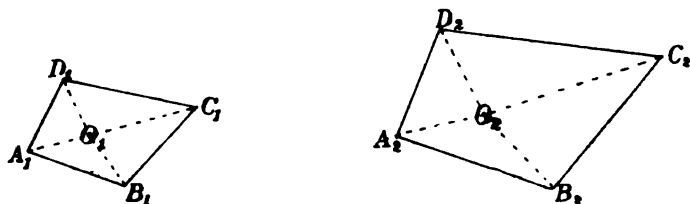


FIG. 3.

Then

$$\begin{aligned} \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2} &= \frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}; \\ \frac{\triangle B_1O_1C_1}{\triangle B_2O_2C_2} &= \frac{\text{square on } B_1C_1}{\text{square on } B_2C_2} \\ &= \frac{\text{square on } A_1B_1}{\text{square on } A_2B_2} \\ &= \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}. \end{aligned}$$

Similarly $\frac{\triangle C_1O_1D_1}{\triangle C_2O_2D_2}$ and $\frac{\triangle D_1O_1A_1}{\triangle D_2O_2A_2}$ each $= \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2}$;

$$\therefore \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2} = \frac{\triangle B_1O_1C_1}{\triangle B_2O_2C_2} = \frac{\triangle C_1O_1D_1}{\triangle C_2O_2D_2} = \frac{\triangle D_1O_1A_1}{\triangle D_2O_2A_2}.$$

Hence each of these ratios

$$\begin{aligned} &= \frac{\triangle A_1O_1B_1 + \triangle B_1O_1C_1 + \triangle C_1O_1D_1 + \triangle D_1O_1A_1}{\triangle A_2O_2B_2 + \triangle B_2O_2C_2 + \triangle C_2O_2D_2 + \triangle D_2O_2A_2} \\ &= \frac{\text{figure } A_1B_1C_1D_1}{\text{figure } A_2B_2C_2D_2}; \\ \therefore \frac{\text{figure } A_1B_1C_1D_1}{\text{figure } A_2B_2C_2D_2} &= \frac{\triangle A_1O_1B_1}{\triangle A_2O_2B_2} \\ &= \frac{\text{square on } A_1B_1}{\text{square on } A_2B_2}. \end{aligned}$$

Euc. VI. 22 follows from the above immediately.

In this way all necessity for using Euclid's phraseology involving the Compounding of Ratios and Duplicate Ratio is removed. At the same time I am not without hope that a reaction in favour of using the ideas of Euclid's Fifth Book may set in. There is this incidental advantage, that the frequent employment of what it is now usual to call Archimedes' Axiom (though Euclid used this axiom freely in his Fifth Book), which is involved in Euclid's line of treatment, draws attention to an idea of the greatest value. Its employment is involved in the proper treatment of magnitudes which exceed any given magnitude however great and magnitudes which are less than any given magnitude however small.

On this axiom for example depend

(A) The proofs of the algebraical propositions—

(1) that if $a > 1$: then it is possible to find n so great that a^n shall exceed any magnitude k however large; expressed usually $\lim_{n \rightarrow +\infty} La^n = +\infty$;

*(2) that if $0 < a < 1$; then it is possible to find n so great that a^n shall lie between 0 and ϵ , where ϵ is any positive magnitude however small; expressed usually $\lim_{n \rightarrow +\infty} La^n = 0$.

(B) The proof of the geometrical proposition that points on a hyperbola exist whose distances from a focus exceed any length however great.

In this way the methods of the Fifth Book form an introduction to the Infinitesimal Calculus of the greatest value. M. J. M. HILL.

UNIVERSITY COLLEGE, LONDON.

ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION.

The annual meeting of the Mathematical Association was held on Saturday, at King's College, London. In the absence of Mr. J. Fletcher Moulton, K.C., M.P., the retiring president, Professor A. Lodge was in the chair, and among those present were Professor A. R. Forsyth, Mr. F. W. Hill (treasurer), Mr. C. Pendlebury (secretary), Mr. A. W. Siddons, Mr. H. D. Ellis, Mr. W. H. Hudson, Mr. James Wilson, Mr. W. N. Roseveare, and Dr. F. S. Macaulay. The report of the council, which was adopted, stated that the association consisted of 351 members, and it referred to the work done by the committee appointed by the association to consider the subject of the teaching of elementary mathematics.

Professor Forsyth was elected president for the forthcoming year, and afterwards presided over the meeting. He remarked that the teaching of elementary mathematics had been a great deal under their consideration, and under the consideration of the local examination syndicate at Cambridge and that of the general syndicate of the whole University. It was not, however, for him to forecast what might be the issue of the deliberations upon the subject; but, as they probably knew, the local examination syndicate had modified their examinations and subsequent changes had been made in the schedule of mathematics. It was his privilege to serve on two of those bodies, and as a member of the committee that worked towards the modification of the regulations of the local examinations he wished to pay a tribute to the work done by the association.

* Although this proposition is necessary for the summation of an infinite number of terms of a Geometrical Progression with the common ratio less than unity, I believe there is only one English Text Book (Chrystal's *Algebra*) which contains a demonstration, and unfortunately that is far more difficult than it need be.

† Euclid proves this in the first proposition of the Tenth Book, for

$$0 < a \leq \frac{1}{2}.$$

Mr. Siddons (Harrow School) submitted the report of the committee on the teaching of elementary mathematics, which, he said, had been criticised as very conservative. With regard to what had been done by various examining bodies, he remarked that in most cases their recommendations had been followed. University bodies were now alive to the necessity of not issuing regulations which clashed with one another. He thought it would be disastrous if that occurred, but he did not believe that there was any danger of that. The most immediate need was that the preparatory schools should move in the matter, and they should get the headmasters of such schools to adopt a more modern treatment of mathematics. It would not be done in the public schools unless the boys were taught from the beginning.

In a short discussion which followed the President said he regarded the teaching of the theory of incommensurables as a most advanced University subject, and the idea of attempting to teach the theory to the average school-boy seemed to him to be almost an impossibility. He agreed with the line adopted by the committee in dealing with the report. It was desirable that they should not hurry changes. It did not lie with the public schools or the preparatory schools to make changes. They had a vast body of teachers in the small schools, but the great difficulty was to get at such teachers and induce them to change.

The report was adopted. Papers were afterwards read on "Some Class Diagrams for Intuitional Geometry" by Mr. E. M. Langley, on "The Representation of Imaginary Points on a Plane by Real Points" by Professor A. Lodge, and on "Incommensurables by Means of Continuous Decimals" by Mr. Edwin Budden.

MATHEMATICAL NOTES.

120. [v. 1. a.] *The Teaching of Arithmetic.*

The following answer, of which I have seen many in examination papers, is instructive.

Q.—Find to four places of decimals the value of $100000 \div 123456$ (123456

$1 + \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \text{etc. when } x = 10.$	$\begin{array}{r} 100000 \\ 123456 \\ \hline 234560 \\ 200000 \\ \hline 345600 \\ 300000 \\ \hline 456000 \\ 400000 \\ \hline 560000 \\ 500000 \\ \hline 600000 \\ 600000 \\ \hline \end{array}$
$A. - 1 + \frac{2}{10} + \frac{3}{100} + \frac{4}{1000} + \frac{5}{10000} + \frac{6}{100000}$ $= \frac{100000 + 20000 + 3000 + 400 + 50 + 6}{100000}$ $= \frac{123456}{100000}$	
Ans.—123456.	

G. H. BRYAN.

121. [L¹ 12. c.] *Isogonal Transformation.*

To prove 1 conic can be drawn through 5 pts.(1)
 2 conics " " " 4 pts. and touching 1 str. line. (2)
 4 " " " " 3 pts. " 2 str. lines. (3)
 4 " " " " 2 pts. " 3 " (4)
 2 " " " " 1 pt. " 4 " (5)
 1 conic " " touching 5 str. lines.(6)

(1) Let A, B, C, D, E be the 5 pts. By isogonal transformation with regard to ABC as many conics can be drawn through A, B, C, D, E as str. lines through D', E' the isogonals of D, E , i.e. 1.

(2) Let A, B, C, D be the 4 pts. and q the str. line. Transform with regard to A, B, C . Then as many conics can be drawn to pass through A, B, C, D and touch q as str. lines can be drawn to pass through D' and touch conic q' , the isogonals of D and q , i.e. 2.

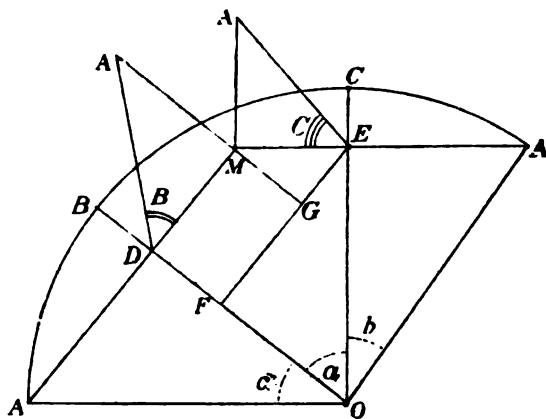
(3) Let A, B, C be the 3 given pts. and q, r the two str. lines. Transform with regard to A, B, C . Then as many conics can be drawn through A, B, C to touch q, r as str. lines can be drawn to touch conics q', r' the isogonals of q, r , i.e. 4.

(4), (5) and (6) follow from (3), (2), (1) by reciprocation.

H. L. TRACHTENBERG.

122. [K. 20. f.] *The Fundamental Formulae of Spherical Trigonometry.*

The proof, of which an outline note is subjoined, is a slight modification of one given by Mr. W. W. Lane, R.N. in his *Spherical Trigonometry*.* The figure given below possesses the advantage of shewing all lines in their true lengths and angles of their true magnitude. Paper models obtained by cutting out $OABCAO$ or $OADAMAEAO$ and folding them up, are very easily made. The formulae for a right-angled spherical triangle are readily deduced in a similar manner.



Let ABC be a spherical triangle; O the centre of the sphere.

Draw AD, AE, AM perpendicular to OB, OC and the plane BOC respectively.

Join MD, ME . Then

$$\begin{aligned} MD^2 &= AD^2 - AM^2 \\ &= OA^2 - OD^2 - (OA^2 - MO^2) \\ &= OM^2 - OD^2. \end{aligned}$$

*[These proofs are really due to our constant and most valued contributor, Mr. R. F. Davis, who discovered them in 1874, while an Undergraduate at Queen's College, Cambridge. Mr. Davis published them in the *Messenger of Mathematics*, Vol. IV., p. 102, and in the *Gazette*, Vol. I., p. 40. It is curious that such simple proofs have not found their way into general use. It is perhaps worth noting that they will be found in Sir Robert Ball's forthcoming *Spherical Astronomy*.]

In 1878 Isaac Todhunter characterised the proofs as "interesting and elegant," and intended to introduce them into a new edition of his *Spherical Trigonometry*. J. Casey in 1888 wrote of "these beautiful proofs" that "they are certainly new to me, and I have no doubt they are original." He also expressed his intention to insert them in an Appendix to his *Spherical Trigonometry*. We are glad to have the opportunity of once more drawing attention to them.

W. J. G.]

Therefore MD is perpendicular to OB . Similarly ME is perpendicular to OC . Let the triangles ADO , AMD , AME , AEO revolve about OD , DM , ME and EO respectively, into the plane BOC . We thus obtain the annexed figure, and identify a , b , c , B , C therein.

Draw EF perpendicular to OD and MG perpendicular to EF .

We have

$$OD = OF + FD, \text{ or, } R \text{ being the radius } OA,$$

$$R \cos c = R \cos b \cos a + R \sin b \cos C \sin a \quad (FD = MG),$$

i.e.

$$\cos c = \cos a \cos b + \sin a \sin b \cos C. \dots\dots\dots(1)$$

Again

$$EF = FG + GE, \text{ that is}$$

$$R \cos b \sin a = R \sin b \sin C \cot B + R \sin b \cos C \cos a;$$

or, dividing by $R \sin b$

$$\cot b \sin a = \cot B \sin C + \cos C \cos a. \dots\dots\dots(2)$$

Again

$$R \sin b \sin C = AM = R \sin c \sin B;$$

$$\therefore \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \dots\dots\dots(3)$$

The modification of the above figure when b or c is greater than 90° presents no difficulty. C. S. JACKSON.

123. [L¹ 1. a.] Note on the Parabola.

I. The general equation to the parabola must be of the form

$$(x-A)^2 + (y-B)^2 = (Lx + My + N)^2 \{L^2 + M^2\},$$

where A , B are the coordinates of the focus and $Lx + My + N = 0$ is the equation of the directrix.

The above equation becomes by transposition $(Mx - Ly)^2 + \dots = 0$, showing that the terms of the second degree form a perfect square and represent a line through the origin perpendicular to the directrix, that is, the diameter through the origin.

II. To reduce the general equation $(ax + by)^2 + 2gx + 2fy + c = 0$ to this form.

Assume $\beta x - \alpha y + \kappa = 0$ as the equation to the directrix, where κ is at present undetermined and has to be found. Then

$$(a^2 + \beta^2)(x^2 + y^2) + 2x(g + \beta\kappa) + 2y(f - \alpha\kappa) + c + \kappa^2 = (\beta x - \alpha y + \kappa)^2,$$

and coordinates of focus are

$$-(g + \beta\kappa) / (a^2 + \beta^2),$$

$$-(f - \alpha\kappa) / (a^2 + \beta^2).$$

Also

$$(g + \beta\kappa)^2 + (f - \alpha\kappa)^2 = (a^2 + \beta^2)(c + \kappa^2),$$

$$\kappa = \{f^2 + g^2 - c(a^2 + \beta^2)\} / 2(\alpha f - \beta g).$$

Thus the focus and directrix are determined.

III. The axis is a line through the focus parallel to the diameter through the origin

$$a \left(x + \frac{g + \beta\kappa}{a^2 + \beta^2} \right) + \beta \left(y + \frac{f - \alpha\kappa}{a^2 + \beta^2} \right) = 0.$$

$$ax + \beta y = (ag + \beta f) / (a^2 + \beta^2).$$

IV. The latus rectum is twice the perpendicular from the focus on the directrix

$$= 2 \frac{-\beta(g + \beta\kappa) + a(f - \alpha\kappa) + \kappa(a^2 + \beta^2)}{(a^2 + \beta^2)^{\frac{3}{2}}} = 2 \frac{\alpha f - \beta g}{(a^2 + \beta^2)^{\frac{3}{2}}}.$$

R. F. DAVIS.

REVIEWS.

Theorie der algebraischen Funktionen einer Variablen und ihre Anwendung auf algebraische Kurven und Abelsche Integrale. By K. HENSEL and G. LANDSBERG. Pp. xvi, 708. (Leipzig: Teubner.)

It is now more than twenty years since the publication in *Crelle's Journal* (vol. 92) of the memoir by Dedekind and Weber, in which it was shown how the theory of algebraic functions and their integrals could be based upon a strictly analytical foundation, without any appeal to geometrical intuition, or the premature introduction of transcendental functions. That paper practically concludes with the establishment of the Riemann-Roch theorem; and, so far as I am aware, the authors have never published the continuation of it which they intended. In any case, it is a great boon to have at last a treatise in which these ideas and methods are fully explained and developed.

The work is divided by the authors into six parts. Of these, the first deals with the mapping out of the values (u, z) which satisfy the equation $f(u, z) = 0$, z being the independent variable. The Riemann sphere is employed, naturally, as a help to the reader; but the facts are established by the method of Puiseux, which is explained in a very thorough and lucid manner. It is also shown how the values of a function are connected by analytical continuation.

The second section deals with the fundamental properties of a corpus; norms, bases, etc., are defined, and, above all, the notion of prime divisors (in Dedekind's sense) is introduced. It is this, with the use of the corresponding notation and terminology, which is one of the most conspicuous features of the work. The notion is a difficult one, as it corresponds to that of an ideal primer in arithmetic: a prime divisor *may* be a function in the corpus, but it need not be. It is always, however, representable as the greatest common measure (in the proper sense) of two functions in the corpus; and this is perhaps the best way of thinking of it as an entity. To fully appreciate this notion of divisor requires time and patience; but when it has been grasped, its extreme value becomes manifest. In my opinion almost every page of this treatise, subsequent to the definition of divisor, bears witness to the precision and simplicity which is gained by adopting Dedekind's principles. The fact that the work is dedicated to Professor Dedekind tends to show that the authors are of the same opinion. Those of us who have been wondering for years why so great a genius has been so inadequately recognised may now hope for better things.

Section III. develops the theory of divisors, and terminates with the proof of the Riemann-Roch theorem. The fundamental problem of this part of the subject is the determination of all the divisors which are multiples of a given divisor. This is effected by the method of ideals. Another important problem is the determination, as a linear aggregate, or "Schaar," of all the integral divisors in a given class. This immediately leads us to the proof of the existence of p linearly independent Abelian differentials, and of the three normal types of Abelian integrals. The Riemann-Roch theorem follows very simply: in fact, almost as a corollary.

Section IV. deals with algebraic structures considered as curves. It is, in fact, an extremely valuable discussion of Plücker's equations in their strict and most general form. Besides this, we have an outline of residuation, and of the theory of adjoint curves.

Section V. treats of correspondence and transformation, and in particular of the "classes" of algebraic structures, and their moduli. Two structures are here taken as of the same class, when they define the same algebraic corpus.

Section VI. is on algebraic relations satisfied by Abelian integrals. This, it is hardly necessary to say, introduces the idea of periodicity, the theorems on interchange of argument and parameter, Abel's theorem, and the problem of inversion. With the statement of the last, the 37th lecture concludes. The 38th and last lecture gives a most interesting account of the historical development of the theory, and of the different ways in which it has been treated.

A reviewer's verdict on a work like this is very likely to be influenced by his own idea of the proper way of treating the subject. Personally, I prefer the method of Dedekind and Weber to any other that I have seen, and value this

treatise accordingly. But apart from the particular method of presentation, the book contains an immense amount of most interesting matter, expounded in a very clear and attractive way. To show that this is not merely my own impression, I may say that last term I lectured on the subject to a young graduate, following the lines of the book, and covering the first twenty lectures or so. The result was quite satisfactory; and the gentleman in question, of his own accord, expressed the opinion that the book was very clear.

I feel sure that others, as well as myself, will be very grateful to Drs. Hensel and Landsberg for providing them with such an excellent treatise on this important and fascinating subject.

G. B. MATHEWS.

Dynamics of Rotation. An Elementary Introduction to Rigid Dynamics. By A. M. WORTHINGTON, C.B., M.A., F.R.S. (London: Longmans, Green & Co.) Fourth Edition. 1902. Pp. 164.

It is a misfortune of most courses on elementary dynamics that the problems which they include are in the majority of cases *unreal*. The "duffer" who cannot see what use it is to find the acceleration of a *perfectly smooth particle* on an inclined plane is certainly worthy of more consideration than the pet student who describes glibly how the acceleration of gravity may be found by Atwood's machine, using the formula $g(P-Q)/(P+Q)$, and finishing up with "Q.E.D.," when the method is fundamentally incorrect.

The dynamics of rotation of a rigid body ought to receive a more prominent place than has hitherto been accorded it, not only in the curriculum of students of physics and engineering, for which this book is specially written, but also in every course of mathematics. Many mathematicians have found considerable difficulty in beginning Dr. Routh's treatise for want of some grasp of the fundamental ideas involved in the subject, which become obscured when associated at the outset with too many higher analytical methods.

Prof. Worthington's book seems admirably adapted to give its readers a good insight into the principles of elementary rigid dynamics. There is no reason why the formulae for uniformly accelerated angular motion, $\omega = \omega_0 + At$, $\theta = \omega_0 t + \frac{1}{2} At^2$, and $\omega^2 = \omega_0^2 + 2A\theta$, should not be taken immediately after, or concurrently with, the corresponding properties of linear motion, and if the time commonly spent in discussing elliptic motion about a focus along with the corresponding motion of the hodogram were expended on a study of this book much would be gained.

Whether the word *torque* is really a necessary innovation is a point about which opinions may differ. On the other hand, if the term is used generally, it is hardly as clear as it might be why in some places the author talks of a *couple*. Neither is it very clear how the fundamental property of moments, "Torques are found to be equal when the products of the force and the distance of its line of action from the axis are equal," may be "deduced from Newton's Laws of Motion" (p. 8). On the other hand the construction of "inertia skeletons" showing the moments of inertia of a body, and the positions of its principal axes, and the sections dealing with "centrifugal couples" are good features. The "slug" as unit of inertia may with advantage perish in common with the velo, celo, and tonal. Why is it that whenever it comes to teaching dynamics people will invent a lot of artificial units, and ignore the simple methods which beginners learn in connection with men mowing acres of grass or cats killing mice?

G. H. BRYAN.

I Gruppi Continui di Trasformazioni. By ERNESTO PASCAL. Pp. xi., 358. Milan. (Manuali Hoepli.) 1903.

No idea is now more to the front in Pure Mathematics than that of a group. Discrete groups, and in particular groups of substitutions, pervade recent algebraical and arithmetical work: transformation groups are ever under consideration in continuous analysis and differential geometry. The conception of the latter class of groups, the subject-matter of the book before us, is the more recent. It eluded the grasp of the discoverers of facts of invariance, for the thorough appreciation of which it was needed. Sophus Lie first formulated it in 1871; and to his genius is due a marvellously full development of the theory.

If we are given n relations expressing n letters $x'_1, x'_2, \dots, x'_n$ —for brevity let us say n letters x' —as functions of n letters x and r parameters a , which are soluble without ambiguity for the letters x , we have a transformation, one between the x

and the x' . If we take another set of n relations, expressing n letters x'' as functions of the n letters x' and r parameters b , we have a second transformation, one between the x' and the x'' . If between the two sets of relations we eliminate the letters x' , we obtain n relations expressing the x'' as functions of the x and the two sets of parameters a, b —a transformation between the x and the x'' , which, arising as it does from the sequence of the two former transformations, is called the product of the two in the order given. Order is not immaterial.

Now let the sets of n functional forms in the x, x' and x', x'' transformations be the same. The n functional forms in the resultant x, x'' transformation will, as a rule, differ from them. But there exist classes of cases in which the old functional forms are reproduced, with r functions of the parameters a, b for new parameters. In such a case the transformations, for all admissible values of the parameters, form a *group*, a *continuous group* as it is supposed that the parameters may be varied continuously.

This continuity, together with the group idea, suggests procedure in transformation by infinitesimal stages, and Lie makes the use of infinitesimal transformation fundamental in his researches. By its means the performance even of a finite transformation of a *group* is reduced to a matter of differential operation. An infinitesimal transformation gives to a function ϕ of the letters x an increment $\delta t X\phi$, where δt is infinitesimal and X is an operator linear in symbols of partial differentiation; and a finite transformation of the one-parameter group which the infinitesimal transformation specifies is performed by operation with

$$1 + tX + \frac{t^2}{1.2} X^2 + \dots,$$

which may be written e^{tX} .

To those who like ourselves have been trained to expertness in continuous analysis this study should be inherently attractive. A certain difficulty which some of us have experienced in getting to feel at home in it, is perhaps due to the presence in our minds of an idea, as to what we should have meant by the performance of a transformation, which was not the idea dominant in Lie's mind. The latter idea has, of course, to take possession of us before confusion of thought as we follow Lie's arguments can cease. By performance of the x to x' transformation on a function we have to mean, not the substitution in the function for the letters x of their expressions in terms of the letters x' , but the substitution for the letters x of the letters x' , or, more frequently, of the expressions for these in terms of the letters x . Lie was before all things a geometer. Transformation to him meant, in the first place, rearrangement in space. An x to x' transformation meant transference of points with coordinates x to the places of other points with coordinates x' given as functions of the x .

Our danger of confusion is greatest when we deal with order in products of transformations. Taking formulae of successive transformation $x' = f(x)$, $x'' = F(x')$, the passage from $\phi(x)$ to $\phi\{F[f(x)]\}$ is in Lie's theory a result of the first transformation followed by the second—a substitution first of x' for x and then of x'' for x' . The equally reasonable observation from a different point of view, that it would result from first replacing x by $F(x)$ according to the x'', x' relationship, and then replacing x by $f(x)$ according to the x', x relationship, is kept in the background. A lesson may however be drawn from this reversal of the order of ideas. We are in fact told that, if S denote the operation of transforming x to x' or $f(x)$, and T that of transforming x to $F(x)$, so that T' properly means that of transforming x' to $F(x')$ or x'' , we must have

$$T'S = ST,$$

where, as in the book under review, the first performed operation in a product is written on the right. We shall encounter presently an example of extrication from temporary confusion by means of a case of this theorem.

The little book on Lie's theory which is before us deserves a hearty welcome. For a short time longer there is still no English book on the subject. Let those of us who know a little Italian peruse the present manual. It is all the easier to start upon because there is not room in it for the dignified style and the almost wearisome elaboration of the greater works brought out under Lie's own auspices. Few authors know so well as Sig. Pascal how to present higher mathematics in didactic form. The range of his mathematical learning is moreover cyclopedic.

The rate at which one useful and up-to-date *Manuale Hoepli* from his pen follows another is remarkable. Signs of haste in production, though not entirely absent, are rare.

The *Manuali Hoepli* are books of size for the pocket. Two pages would go on one of an ordinary octavo. The type, which is beautifully clear, is almost extravagantly large. The purchaser for half-a-crown of the present volume might well be forgiven for expecting only a meagre sketch of first principles, and not much of the analysis, perforce abounding in triple suffixes, etc., which is formidable of aspect even on the ample page of Teubner's *Lie-Engel*. He will be agreeably disappointed. The work is not unambitious. Its aim is "Without lack of rigour and generality to contain in little space all that forms the basis of this advanced part of pure mathematics." The author has at any rate succeeded in making clear in their complete forms the principles and processes of the general theory. Special theories, and in particular the whole subject of contact transformations, are reserved for a promised further volume.

Lie is of course followed closely, with judicious and lucid condensation. Essentials are only omitted from arguments in certain lengthy and elaborate investigations which are honestly presented as given in sketch only, or even, where fundamentals are not at stake, by mere reference. Expectation of more than Lie gives is at first raised by the mention of groups which are not "Gruppi di Lie," Lie having confined attention to groups which contain the identical transformation and of which all other transformations go in inverse pairs, whereas there may be groups which, for such reasons as that their parameters have not all values open to them, do not possess these properties. But groups other than Gruppi di Lie are not in the sequel subjected to analytical treatment.

The one important addition to Lie's analysis which occurs is a study (pp. 84, etc.) of the product of two distinct transformations each of which is a finite transformation of some one-parameter group, with matter leading up to the study, and some application of its conclusions. Proofs based on it are as a rule only given as alternative to Lie's proofs. In this study signs of haste are unfortunately noticeable, and the reader is hardly left convinced. It is not, however, now intended to throw doubt on the conclusions. There is support of which the author is not conscious. He is mistaken in believing that the matter was first dealt with by himself in 1901. He will find substantially the same theory and essentially the same conclusions in two papers, by Mr. J. E. Campbell "On a law of combination of operators bearing on the theory of continuous transformation groups," in Vols. XXVIII. and XXIX (1897) of the *Proceedings of the London Mathematical Society*.

An existence theorem is really in question in the study referred to. An x to x' transformation is performed by operation with e^{X_1} , and then an x' to x'' transformation by operation with e^{X_2} . X_2 is the formal result of replacing accented letters by unaccented in X_1 . The existence theorem is that a linear operator Z exists, and is linear with constant coefficients in tX_1 , $t'X_2$, and a succession of alternants ("parentesi di Poisson") derived from these, which is such that the resultant x to x'' transformation can be performed by operation with e^Z , i.e. that

$$e^{X_2} e^{X_1} = e^Z.$$

That the theorem in question is not stated by our author. It is even doubtful whether he has realized the fact. The existence of a linear Z , of some form or other, he appears to assume in the one word "naturalmente." It almost looks as if he took the operational equality just written to mean only that the product of two transformations is a transformation, whereas it really means that the product of two transformations each of which belongs to some group generated by an infinitesimal transformation is a third transformation with a like property. The proof of the existence of Z is left to depend on the actual discovery of the coefficients in a Z which suffices. This being so it is to be regretted that space could not be found for fuller proof of success in finding them.

There is another hasty misconception in the text, at the outset of the same investigation, which happily the author discovered in time to correct it in a note at the end of the book. The error was to suppose that in $e^{X_2} e^{X_1}$ we may without change of meaning replace X_2 by X_1 . The correction is by means of a change of notation and a proof of what is practically the theorem of reversal of order

$$e^{X_2} e^{X_1} = e^{X_1} e^{X_2},$$

which is a particular case of one arrived at just now. The immediate consequence of this

$$e^{x_1} e^{x_2} = e^{x_1 + x_2},$$

and the more general

$$T = STS^{-1},$$

are in interestingly close analogy with well-known facts as to the transformation of one substitution by another.

One of the many notes at the end of the volume contains a new proof, not given in full detail, of the difficult second part of Lie's third fundamental theorem.

E. B. ELLIOTT.

A Text-book of Field Astronomy for Engineers. By GEORGE C. COMSTOCK, Director of the Washburn Observatory, Professor of Astronomy in the University of Wisconsin. Pp. x, 202. (John Wiley & Sons, New York.)

This work consists of nine chapters, entitled: Introductory, Coordinates, Time, Corrections to coordinates, Rough determinations, Approximate determinations, Instruments, Accurate determinations, and The transit instrument. As its name implies, the book is intended for engineering students, and the "unconventional views" contained therein have developed during many years' experience in teaching the elements of practical astronomy. The "unconventional views" are excellent things. In the reviewer's experience the conventional view of a transit instrument from a student's stand-point is that three errors are connected with it; these he is quite eager to explain, but he is rather inclined to believe that the instrument is of no manner of use, and he is rather insulted if one asks him what it is for. Therefore, let those who teach and examine in astronomy, and who have no instruments to look after and no observations to make, get this book and acquire unconventional views as speedily as possible. The book will be found interesting and useful to the numerous amateurs who take pleasure in making time observations, etc. We have tried to show that the work will be useful and interesting to persons for whom it was not written, and we wish it to be inferred *a fortiori* that the engineer will find in it all he wants—and he often wants what he cannot find in the ordinary text-book on astronomy.

C. J. JOLY.

Specielle Algebraische und Transcendente Ebene Kurven. By GINO LORIA. Translated into German by FRITZ SCHUTTE. 2 vols., pp. xiv, 744, with 174 figures.

It may be admitted with considerable truth that few exercises are more useful to the student than the tracing of a number of curves. It is imperative that before the pupil is introduced to the general theory of curves he shall be familiarized with asymptotes, nodes, cusps, and the like, by the detailed discussion of numerical equations. The study of curves has its practical value in Mechanics and Physics, and we ought not to ignore its aesthetic value. Beauty and elegance, and even the "wild civility" of the asymmetric curves, have their attractions to the human young. Curves, moreover, appeal to the historic sense as perhaps does no other branch of science. The first real impulse which awakened the study of this department of Mathematics out of the lethargy from which it had suffered since the days of the Greeks was given by Descartes. Then came a period of development in which figure the names of Cavalieri, Wallis, Roberval, Pascal, and Newton. In reading the works of these founders of a new school, one has to be wary. Lack of communication, and the rarity and costliness of literature made it inevitable that there should be many working in the same field who were unknown to each other. Almost the last piece of work done by Pascal was the discussion of a curve to which he had given the name of the roulette. The same curve was dealt with by Roberval under the name of the trochoid; to this generation it is familiar under the title of the cycloid, and indeed it was known to Galileo under that name. The historical sense of the school-boy may in this case be tickled by the knowledge that Pascal was suffering from the toothache and insomnia when the thought of his "roulette" entered his mind. The disappearance of all aches and pains shortly after he had begun to brood over the subject was piously regarded by him as an intimation from above, that the Great Architect of the Universe was viewing with unqualified approval the attack of the problems which had exercised the ingenuity of Galileo when dealing with rolling curves in connection with the construction of the arches

of bridges. Most teachers may have noticed that it is comparatively easy to excite an interest in the personality of the inventor of a theorem or a law. Who would not prefer the name "Euler line" to "CONG line" for example? To study a curve in detail, to see how the various properties are brought to light, to realise how the history of the curve has been influenced by the progress of discoveries which perhaps had nothing to do with the art of curve tracing, to see where and why one man failed and his friend and rival, it may be, succeeded—this cannot but prove a stimulus to the young and enquiring mind, and may prove the germ of what in later years will ripen into efflorescence. But for even a short lecture such as is here suggested the teacher cannot rely for information on the ordinary text-books. Take for instance Frost's *Curve Tracing*. It contains, we verily believe, but two names in all its two hundred pages—Newton and De Gua. Think how much more interesting that admirable work might have been made by even the most elementary references to the history of the art. The teacher has no longer the excuse that he does not know whence to draw his material. This translation into German of Dr. Loria's encyclopaedic work will be far more than is necessary for such a modest programme as is permitted to the teacher when other claims are considered. It is monumental. From cover to cover it teems with interest. A copy should be on the shelves of every College Library, for "the sight of means to do good deeds make good deeds done." The student who wishes to carry further his researches will find all that he requires in the shape of references from the literature of the earliest days up to the most recent memoirs. The amount of patient labour which these two volumes with their 750 pages represent is colossal.

The Foundations of Geometry, by D. HILBERT. Authorised translation, by E. J. TOWNSEND. Pp. viii, 142. 4s. 6d. net. (Open Court Co.: Kegan Paul.) 1902.

A translation of Hilbert's fascinating *Grundlagen der Geometrie* is heartily welcome in this country, and the volume under notice is further enriched by the author's additions, which appeared in the French translation which M. Laugel published some years ago (Gauthier-Villars). It also contains a summary of a memoir embodying Hilbert's latest researches, which has probably already appeared in the *Math. Ann.*

The first attempt to prove the concurrence in the plane of lines which are not parallel was made by Legendre. He showed that if any one triangle has the sum of its angles equal to two right angles, then the sum of the angles of all triangles will be two right angles; but he failed in his endeavour to prove the existence of one such triangle. Saccheri in 1733, and later Gauss, Lobatchewsky, and Bolyai attacked the same problem, but on different lines. They started with the negation of the axiom of parallels, and to the great surprise and alarm of Saccheri (v. Russell, *Foundations of Geometry*, p. 8) the result was more than one Geometry to the logical basis of which no objection could be found. Their success led to further investigations as to the axioms in general. The conception of space as a manifold of numbers gave Riemann, Helmholtz, and Lie the opportunity of establishing on an analytical basis both the non-Euclidean system of Lobatchewsky, and the system in which Euclid's "straight line" is avoided. In the former the sum of the angles of a triangle is always less, and in the latter always greater, than two right angles. On the other hand, we have the purely geometrical investigations of Veronese and Hilbert. How then are the researches of Hilbert to be placed with reference to the analytical researches of other workers in the same field? Helmholtz showed that Euclid's propositions were in disguise but the laws of motion of solid bodies. The non-Euclidean propositions were in the same manner the laws to which are subject bodies analogous to solid bodies, but with no physical existence. Lie went further. Combining all the possible transformations of a figure he calls the total a group. To each of these groups he attached a geometry; all these geometries have common properties; but the generality of his conclusions is impaired by the fact that all his groups are continuous. His space is a *Zahlenmannigfaltigkeit*. His geometries are subject to the forms of analysis and arithmetic. Now, as M. Poincaré points out (*Bull. des Sciences Mathématiques*, Sept. 1902), this is exactly where Hilbert comes in. His spaces are not *Zahlenmannigfaltigkeiten*. The objects he calls point, line, or plane are purely

logical conceptions. The most important of the additions to be found in the French translation are due to the investigations of Dr. Dehn, of which mention has been made already in our columns (No. 29, Oct. 1901, p. 94). The geometry which he constructs is one in which the sum of the angles of any triangle is two right angles; in which similar non-congruent triangles exist; and in which an infinite number of parallels to any straight line may be drawn through any point.

As a translation the volume before us cannot be said to be entirely successful. It has been unmercifully and somewhat undeservedly gibbeted by Prof. Halsted in *Science*, Aug. 22, 1902. A sober and detailed criticism by Dr. Hedrick of both this and the French translation will be found in the *Bull. of the American Math. Soc.*, Dec. 1902, to which considerations of space compel us to refer the reader, and in which will be found a long list of errors and misprints. We have carefully compared the French and English translations, and we find that Dr. Hedrick has omitted no point of any consequence, and that in our opinion his strictures are quite justified and necessary to the clear understanding of the text. With his list of errata the translation in question may be read by the student who can appeal to an expert for guidance, but on the whole we should prefer to place M. Laugel's book in his hands.

PROBLEMS.

469. [K. 20. b.] If ϵ be so small that its cube and higher powers may be neglected, and $\phi - \theta + 2\epsilon \sin \theta = \frac{3\epsilon^2}{4} \sin 2\theta$, then $\theta - \phi - 2\epsilon \sin \phi = \frac{5\epsilon^2}{4} \sin 2\phi$.

ANON.

470. [D. 2. b.] If successive terms of a series are connected by the relation $u_n u_{n-3} - u_{n-1} u_{n-2} = 2a$, and the first three terms are unity, prove that (i.) all the terms are integral functions of a , (ii.) any even term is the arithmetic mean of the adjacent terms; and find an expression for the n th term.

W. H. LAVERY.

471. [K. 10. c.] Prove that the general form of the approximation to the length L of a circular arc when powers of $\frac{L}{R}$ beyond the $2(n+1)$ th are neglected, R being the radius of the circle, is given by L_n where

$$L_n a_1 a_2 a_3 \dots a_n = l_0 - l_1 \frac{a_n}{a_1} 2^{b_1} + l_2 \frac{a_n a_{n-1}}{a_1 a_2} 2^{b_2} - \dots + (-1)^n l_n 2^{b_n},$$

where $a_r = 1 - 4^r$; $b_r = r(r+2)$; $l_r = \text{chord of arc } \frac{L}{2^r}$.

(Huygens's approximation is given by $n=1$.)

R. M. MILNE.

472. [L. 17. d.] Given the focus and directrix of a parabola, find by Euclidean methods the points in which the parabola cuts a given straight line. Solve also the same problem for the ellipse.

R. F. MUIRHEAD.

473. [L. 17. e.] A family of conics have their axes along given lines and pass through a given point; show that the locus of the centre of curvature at the given point is a cubic whose asymptotes are all real.

C. F. SANDBERG.

474. [K. 2. e.] In a triangle ABC are inscribed three squares, of which the sides parallel to AB , BC , CA are DE' , EF' , FD' . Prove that the circles AEF' , BFD' , CDE' touch one another, and that the radius of the smaller circle which touches all three is $R/(4 \cot \omega + 7)$.

C. E. YOUNGMAN.

475. [K. 8. a.] The angles α , β , γ , δ of a quadrilateral satisfy the relation

$$\sum \cos^2 \frac{\alpha}{2} + 2\Pi \cos \frac{\alpha}{2} = 2 + 2\Pi \sin \frac{\alpha}{2}$$

Durham, 1902.

476. [B. 3. 4.] Eliminate θ, ϕ from the equations

$$\sin \theta + \sin \phi = a; \quad \sin 2\theta + \sin 2\phi = b; \quad \sin 3\theta + \sin 3\phi = c;$$

and eliminate θ from the equations

$$16 \sin^5 \theta - \sin 5\theta = a; \quad 16 \cos^5 \theta - \cos 5\theta = b. \quad (C.)$$

477. [A. 2. b.] Solve the equations:

$$\begin{aligned} (b-c)x + (c-a)y + (a-b)z &= 0; \\ (b^2-c^2)yz + (c^2-a^2)zx + (a^2-b^2)xy &= 0; \\ (b-c)yz + (c-a)zx + (a-b)xy + (b-c)(c-a)(a-b) &= 0. \end{aligned} \quad (C.)$$

478. [A. 1. b.] If

$$\begin{aligned} \phi(x, y, n) &\equiv \frac{x^n}{n} - x^{n-2}y + \frac{n-3}{2} x^{n-4}y^2 \\ &- \frac{(n-4)(n-5)}{3} x^{n-6}y^3 + \frac{(n-5)(n-6)(n-7)}{4} x^{n-8}y^4 - \dots, \end{aligned}$$

where n is a positive integer, then shall

$$2\phi(\alpha, \beta, 2n) = \phi(\alpha^2 - 2\beta, \beta^2, n). \quad (C.)$$

479. [K. 2. a.] Tangents are drawn to the circum- and 9-pt. circles of a triangle ABC where they are met by the join of A to the orthocentre. Find the area of the quadrilateral thus formed. (C.)

480. [L¹. 17. a.] Two equal and concentric ellipses are inscribed in a quadrilateral. Their axes are at an angle α . Shew that the area of the quadrilateral is

$$2\sqrt{(\alpha^2 - b^2)\sin^2 \alpha + 4a^2b^2}.$$

Two similar concentric ellipses of same eccentricity e , have their major axes at an angle α ; shew how to find α if the foci of the first be on the second, and the ends of the minor axis of the second lie on the first. (C.)

SOLUTIONS.

406. [K. 3. a.] (Corrected.) OAB is an isosceles triangle, base AB , and PAB is any other triangle on the same base and in the same plane: prove that

$$4OA^2 \cdot PA \cdot PB \cdot \cos^2 \frac{1}{2}(AOB - APB) + BA^2 \cdot OP^2 = OA^2(PA + PB)^2. \quad (C.)$$

Solution by R. N. APPE.

Put

$AB=c$; $OA=OB=a$; $PA=x$; $PB=y$; $PO=z$; $\angle APB=\theta$; $\angle APO=\phi$; while α, A denote the angles O, A of the triangle OAB .

(i.) Taking O, P on same side of P ,

$$c^2 = x^2 + y^2 - 2xy \cos \theta,$$

$$a^2 = x^2 + z^2 - 2xz \cos \phi,$$

$$a^2 = y^2 + z^2 - 2yz \cos(\theta + \phi);$$

$$\therefore ac \cos A = \frac{1}{2}(c^2 + a^2 - a^2) = -xy \cos \theta - xz \cos \phi + yz \cos(\theta + \phi) + x^2.$$

Also $ac \sin A = xy \sin \theta + xz \sin \phi - yz \sin(\theta + \phi).$

Rearranging, squaring, and adding, we find

$$x^2(x^2 + y^2 - 2xy \cos \theta)$$

$$= a^2c^2 + x^2y^2 + x^4 + 2acxy \cos(A + \theta) - 2acx^2 \cos A - 2x^2y \cos \theta,$$

$$\text{i.e.} \quad c^2x^2 = a^2c^2 + c^2x^2 + 2acxy \cos(A + \theta) - 2acx^2 \cos A$$

$$= a^2c^2 + 2acxy \cos(A + \theta). \quad (\because c = 2a \cos A.)$$

(ii.) If O, P be on opposite sides of AB ,

$$c^2z^2 = a^2c^2 + 2acxy \cos(A - \theta).$$

$$\therefore c^2z^2 = a^2c^2 + 2acxy \cos(A \pm \theta)$$

$$= a^2(x^2 + y^2 - 2xy \cos \theta) + 2acxy \cos(A \pm \theta)$$

$$= a^2(x + y)^2 - 2xy[a^2(1 + \cos \theta) - ac \cos(A \pm \theta)]$$

$$= a^2(x + y)^2 - 2a^2xy[1 + \cos \theta - 2 \cos^2 A \cos \theta \pm 2 \sin A \cos A \sin \theta]$$

$$= a^2(x + y)^2 - 2a^2xy[1 - \cos(2A \pm \theta)]$$

$$= a^2(x + y)^2 - 2a^2xy[1 + \cos(\alpha \mp \theta)] \quad \left(\because A = \frac{\pi}{2} - \frac{\alpha}{2} \right)$$

$$= a^2(x + y)^2 - 4a^2xy \cos^2 \left(\frac{\alpha \mp \theta}{2} \right),$$

according as O, P are on the same side or on opposite sides of AB .

Solution by R. M. MILNE.

Let C be mid-point of AB . Describe the circumcircle of the triangle APB , and denote the diameter which passes through O by TT' . A and B are the foci of an ellipse which passes through P , and PT, PT' are the tangent and normal at P . In the following analysis the constants a, b, c refer to this ellipse, and P is the point (x, y) referred to the usual axes. The point O will be (o, c) .

Transposing, the question reads

$$4OA^2 \cdot PA \cdot PB \cos^2 \frac{AOB - APB}{2} = OA^2(PA + PB)^2 - AB^2 \cdot OP^2.$$

$$\text{The Left-hand member} = 4 \cdot OA^2 \cdot PA \cdot PB \cos^2(AOC + ATC)$$

$$= 4 \cdot OA^2 \cdot PA \cdot PB \frac{(OC \cdot CT + AC^2)^2}{AO^2 \cdot AT^2}$$

$$= 4 \cdot y \cdot TT' \frac{\left(c \cdot \frac{b^2}{y} + a^2e^2 \right)^2}{TT'^2 \cdot \frac{b^2}{y}}$$

$$= \frac{4}{b^2} (cb^2 + a^2e^2y)^2$$

$$= \frac{4a^2[c + e^2(y - c)]^2}{1 - e^2}.$$

$$\text{The Right-hand member} = 4a^2(c^2 + a^2e^2) - 4a^2e^2 \cdot OP^2$$

$$= 4a^2[c^2 + a^2e^2 - e^2(x^2 + y - c^2)]$$

$$= 4a^2[c^2 + e^2(a^2 - x^2) - e^2(y - c)^2]$$

$$= 4a^2 \left[c^2 + \frac{e^2y^2}{1 - e^2} - e^2(y - c)^2 \right]$$

$$= \frac{4a^2}{1 - e^2} [c^2(1 - e^2) + e^2y^2 - e^2(1 - e^2)(y - c)^2]$$

$$= \frac{4a^2}{1 - e^2} [c^2(1 - e^2) + y^2e^4 + 2yce^2(1 - e^2)]$$

$$= \frac{4a^2}{1 - e^2} [c(1 - e^2) + e^2y]^2$$

$$= \text{Left-hand member.}$$

415. [K. 9. b.] It is required to inscribe in a given circle $x^2 + y^2 = a^2$ a regular heptagon, one of whose angular points lies at $(a, 0)$. Prove that the two rectangular hyperbolas whose centres are at the points $(\frac{1}{2}a, \pm \frac{1}{2}a\sqrt{7})$, whose transverse axes are parallel to Ox , and which pass through the point $(a, 0)$, cut the circle also in the required angular points. [Suggestions for simpler constructions are invited.]

H. W. RICHMOND.

Solution by H. G. BELL.

The rectangular hyperbolas referred to are

$$\left(x - \frac{1}{4}a\right)^2 - \left(y \pm \frac{\sqrt{7}}{4}a\right)^2 = \frac{1}{8}a^2,$$

and these intersect $x^2 + y^2 = a^2$ in points whose abscissae are given by

$$x^2 - \frac{1}{2}ax + \frac{1}{16}a^2 - a^2 + x^2 \pm \frac{\sqrt{7}}{2}a\sqrt{a^2 - x^2} - \frac{7}{16}a^2 = \frac{1}{2}a^2,$$

$$(4x^2 - ax - 3a^2)^2 = 7a^2(a^2 - x^2).$$

Dividing out by factor $x - a$, we get on reduction,

$$8x^3 + 4ax^2 - 4a^2x - a^3 = 0,$$

whose roots are

$$a \cos \frac{2\pi}{7}, \quad a \cos \frac{4\pi}{7}, \quad a \cos \frac{6\pi}{7}.$$

Hence the points of intersection give the vertices of inscribed heptagon.

Solution by R. F. DAVIS.

If in an auxiliary figure we construct the parabola $y^2 = ax$, and the hyperbola $y(y - x + 2a) = a^2$, the ordinate of the point of intersection of these curves nearest the origin is exactly the side of the regular polygon of fourteen sides inscribed in a circle of radius a .

For if λa be that ordinate, it will be found that

$$\lambda^3 - \lambda^2 - 2\lambda + 1 = 0; \text{ or } \lambda = 2 \sin(\pi/14).$$

Solution by PROPOSER.

The rectangular hyperbola

$$(x - \frac{1}{4}a)^2 - (y - \frac{1}{4}a\sqrt{7})^2 = \frac{1}{8}a^2,$$

intersects the circle $x^2 + y^2 = a^2$ at $(a, 0)$, and at three other points. Write the equation of the hyperbola

$$\{x + iy - \frac{1}{4}a(1 + i\sqrt{7})\}^2 + \{x - iy - \frac{1}{4}a(1 - i\sqrt{7})\}^2 = \frac{1}{4}a^2,$$

and substitute $x = a \cos \theta$, $y = a \sin \theta$, $\cos \theta + i \sin \theta = t$; then

$$\{t - \frac{1}{4}(1 + i\sqrt{7})\}^2 + \{t^{-1} - \frac{1}{4}(1 - i\sqrt{7})\}^2 = \frac{1}{4};$$

$$t^2 + t^{-2} - \frac{1}{2}(t + t^{-1}) - \frac{1}{2}i\sqrt{7}(t - t^{-1}) - 1 = 0;$$

either

$$t = 1 \text{ or } t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - 1 - \frac{1}{2}i\sqrt{7}(t^2 + t) = 0.$$

Now if

$$p = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}, \quad p^7 = 1, \text{ and } p^6 + p^5 + p^4 + p^3 + p^2 + p = -1.$$

$$(p + p^3 + p^5) + (p^6 + p^4 + p^2) = -1,$$

$$(p + p^3 + p^5) \times (p^6 + p^4 + p^2) = 3p^7 + p^6 + p^5 + p^4 + p^3 + p^2 + p = 2.$$

Thus $p + p^3 + p^4$ and $p^6 + p^5 + p^3$ are roots of $z^3 + z + 2 = 0$, and

$$p + p^3 + p^4 = \frac{1}{2}(-1 + i\sqrt{7}), \quad p^6 + p^5 + p^3 = \frac{1}{2}(-1 - i\sqrt{7}).$$

Hence the equation whose roots are p, p^3, p^4 is

$$(t - p)(t - p^3)(t - p^4) = 0,$$

$$\text{or} \quad t^3 - t^2(p + p^3 + p^4) + t(p^6 + p^5 + p^3) - 1 = 0,$$

$$\text{or} \quad t^3 + \frac{1}{2}t^2 - \frac{1}{2}t - 1 - \frac{1}{2}i\sqrt{7}(t^2 + t) = 0.$$

$\therefore$ The hyperbola cuts the circle in four points whose vectorial angles measured from Ox are

$$0, \quad \frac{2\pi}{7}, \quad \frac{4\pi}{7}, \quad \frac{8\pi}{7}.$$

418. [M¹. 2. a.] Find on a given straight line two points which shall be homologous in two given similar figures. C. E. YOUNGMAN.

Solution by R. F. DAVIS.

Let P, Q be homologous points in the two figures; O the centre of similitude. Upon the given straight line AB take points X, Y , such that the angle $OXB = OPQ$ and $OYA = OQP$. Then the triangles OPQ, OXY are similar; etc.

419. [R. 9. b.] A, B are two smooth holes in a smooth horizontal table, distance $2a$ apart; a particle of mass M rests on the table midway between A and B , and a particle of mass m hangs beneath the table suspended from M by two equal weightless inextensible strings, each of length $a(1 + \sec \alpha)$, passing through A and B ; a blow J is given to M in a direction perpendicular to AB . Show that if $J^2 > 2Mmga \tan \alpha$, M will oscillate to and fro through a distance $2a \tan \alpha$, but if J^2 is less than this quantity and $= 2Mmga(\tan \alpha - \tan \beta)$, the distance through which M oscillates will be $2a\sqrt{(\sec \alpha - \sec \beta)(\sec \alpha - \sec \beta + 2)}$.

St. John's (C.), 1895.

Solution by W. F. BEARD; R. F. DAVIS; C. V. DURELL.

Let θ be the angle (in the plane of the table) which the upper portion of either string makes at any instant with AB ; ϕ , the angle in the vertical plane which the lower portion makes with AB . Initially $\phi = \alpha$, and the energy of the system is $J^2/2M$. The potential energy when the system is instantaneously at rest is $mga(\tan \alpha - \tan \phi)$, and the kinetic energy $= 0$. Hence

$$mga(\tan \alpha - \tan \phi) = J^2/2M.$$

$$(i.) \text{ If } J^2 = 2Mmga(\tan \alpha - \tan \beta),$$

$$\phi = \beta, \quad \sec \theta = 1 + \sec \alpha - \sec \beta \text{ [from length of string],}$$

$$\text{and } \tan^2 \theta = (\sec \alpha - \sec \beta)^2 + 2(\sec \alpha - \sec \beta),$$

so that M oscillates through a distance

$$2a\sqrt{(\sec \alpha - \sec \beta)(\sec \alpha - \sec \beta + 2)}.$$

(ii.) If $J^2 = 2Mmga \tan \alpha$, $\phi = 0$, and the lower particle comes to rest in the line AB , and M will oscillate through a distance $2a \tan \alpha$.

(iii.) If $J^2 > 2Mmga \tan \alpha$, m strikes the bottom of the table with a finite velocity, and M still oscillates through $2a \tan \alpha$, but the oscillations are not periodic until after an infinite number of impacts unless the table be inelastic.

425. [K. 2. c.] The sides of a triangle ABC are bisected in D, E, F ; shew that on the circle DEF four positions of a point P may be found such that

$$DP \pm EP \pm FP = 0,$$

and that these are the four points where the circle is touched by the incircle and excircles of the triangle ABC .

Shew also that the tangents to the circle at these four points are also tangents of the ellipse which touches each side of ABC at its middle point.

H. W. RICHMOND.

Solution by PROPOSER.

If D, E, F be three points on a circle touched by a second circle at P , and DP, EP, FP cut the second circle in D', E', F' ,

$$DD' : EE' : FF' :: DP : EP : FP.$$

$\therefore$ the tangents from D, E, F to the second circle have their lengths, viz.,

$$\{DD' \cdot DP\}^{\frac{1}{2}}, \{EE' \cdot EP\}^{\frac{1}{2}}, \{FF' \cdot FP\}^{\frac{1}{2}},$$

proportional to DP, EP, FP .

Hence if P be the point of contact of the incircle of ABC and the N.P. circle

$$DP : EP : FP :: \frac{1}{2}(b+c) : \frac{1}{2}(a+c) : \frac{1}{2}(a+b),$$

for the last are the lengths of the tangents from D, E, F to the incircle.

$$\therefore DP \pm EP \pm FP = 0.$$

So for the escribed circles.

Again, if p_1, p_2, p_3 be the perpendiculars from D, E, F on the tangent at P ,

$$p_1 : p_2 : p_3 :: DP^2 : EP^2 : FP^2.$$

$$\therefore \sqrt{p_1} \pm \sqrt{p_2} \pm \sqrt{p_3} = 0.$$

Hence the tangent touches a conic through D, E, F , the tangents at D, E, F being sides of ABC .

450. [K. 20. e.] In any triangle, r_1, r_2, r_3 being the ex-radii, and s the semi-perimeter, shew that

$$\frac{r_2 + r_3}{(s-a)\sin A} = \frac{r_3 + r_1}{(s-b)\sin B} = \frac{r_1 + r_2}{(s-c)\sin C},$$

and give a symmetrical expression for the common value.

E. N. BARISIEN.

Solution by W. F. BEARD ; J. M. CHILD.

$$\begin{aligned} \frac{r_2 + r_3}{(s-a)\sin A} &= \frac{bc}{2(s-a)} \cdot \frac{r_2 + r_3}{\Delta} \\ &= \frac{bc}{2(s-a)} \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= \frac{abc}{2(s-a)(s-b)(s-c)} \\ &= \frac{abcs}{2\Delta^2}. \end{aligned}$$

451. [J. 1. a. a.] If $f(n)$ be the number of permutations of n letters altogether, on condition that no letter is moved more than one place from its original position in a given order, then

$$L_{n=\infty} \frac{f(n)}{f(n+1)} = 2 \sin 18^\circ.$$

E. M. LANGLEY.

Solution by W. F. BEARD.

The number of permutations in which the first letter retains its place is $f(n-1)$: the number in which it is moved is $f(n-2)$,

$$\therefore f(n) = f(n-1) + f(n-2).$$

Also $f(1)=1$, $f(2)=2$; and if

$$(1-x-x^2) = (1-\alpha x)(1-\beta x),$$

α being positive and numerically greater than β , so that

$$\alpha + \beta = 1 : \alpha\beta = -1,$$

$f(n)$ is the coefficient of x^n in the expansion in ascending powers of x of

$$\begin{aligned} \frac{x(1-x)}{1-x-x^2} &= \frac{x}{\alpha-\beta} \left\{ \frac{\alpha-1}{1-\alpha x} - \frac{\beta-1}{1-\beta x} \right\}; \\ \therefore \frac{f(n)}{f(n+1)} &= \frac{(\alpha-1)\alpha^{n-1} - (\beta-1)\beta^{n-1}}{(\alpha-1)\alpha^n - (\beta-1)\beta^n} = \frac{-\beta\alpha^{n-1} + \alpha\beta^{n-1}}{-\beta\alpha^n + \alpha\beta^n} \\ &= \frac{\alpha^{n-2} - \beta^{n-2}}{\alpha^{n-1} - \beta^{n-1}} = \frac{1}{\alpha} \frac{1 - \left(\frac{\beta}{\alpha}\right)^{n-2}}{1 - \left(\frac{\beta}{\alpha}\right)^{n-1}}; \\ \therefore \lim_{n \rightarrow \infty} \frac{f(n)}{f(n+1)} &= \frac{1}{\alpha} \left[\because \left| \frac{\beta}{\alpha} \right| < 1 \right] \\ &= \frac{-1 + \sqrt{5}}{2} = 2 \sin 18^\circ. \end{aligned}$$

Solution by J. M. CHILD.

If $f(n)=r+s$, where r of the permutations end in a_n , and s in a_{n-1} , then

$$f(n+1) = 2r + s,$$

of which r end in a_n , and $r+s$ in a_{n+1} , and

$$f(n+2) = 2(r+s) + r = f(n+1) + f(n).$$

Hence $\frac{f(n)}{f(n+1)}$ is in general the reciprocal of the convergents to the continued fraction

$$1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \dots}}}} \quad \left[= x + 1 + \frac{1}{x} \right]$$

Hence
$$L_{n \rightarrow \infty} \frac{f(n)}{f(n+1)} = \frac{1}{x} = \frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{\sqrt{5}-1}{2} = 2 \sin 18^\circ.$$

453. [L¹. 17. e.] Two conics have a common focus and their directrices are at right angles: what is the condition that their common chord may pass through the common focus?

A. F. VAN DER HEYDEN (Durham, 1902).

Solution by W. F. BEARD.

Let S be the common focus; PQ the common chord; R the intersection of the directrices; e, e' the eccentricities; PM, PM', QN, QN' perpendiculars on the directrices.

$$e \cdot PM = SP = e' \cdot PM' : e \cdot QN = e' \cdot QN';$$

$$\therefore PQ \text{ passes through } R,$$

and if S lies on PQ ,

$$e \cdot SX = e' \cdot SX',$$

i.e. the conics have equal latera recta; this is also easily proved from the polar equations of the conics.

[The semi latus rectum is a harmonic mean between the segments of any focal chord; hence if two conics have a common focus and a common chord passing through that focus, they must have equal latera recta: the directrices need not be at right angles.]

BOOKS, ETC., RECEIVED.

Elementary Geometry. Books I. and II. By W. M. BAKER and A. A. BOURNE, Third edition, revised. Pp. vi, 128. 1903. 1s. 6d. Books I.-III. Third edition, revised. Pp. viii, 214. 2s. 6d. 1903. Books I.-IV. Pp. viii, 272. 3s. 1903. (Bell.)

Il Pitagora. Edited by G. FAZZARI. Dec. 1902, Jan. 1903.

Periodico di Matematica. Edited by G. LAZZERI. Jan., Feb. 1903.

Supplemento al Per. di Mat. Dec. 1902, Jan. 1903.

Gazeta Matematica. Edited by ION IONESCU. Dec. 1902, Jan. 1903.

The American Mathematical Monthly. Edited by B. F. FINKEL and L. E. DICKSON.

Le Matematiche. Edited by C. ALASIA. Nov., Dec. 1902.

Theoretical Geometry for Beginners. By C. H. ALLCOCK. Pp. x, 135. 1s. 6d. 1903. (Macmillan.)

Annals of Mathematics. Edited by ORMOND STONE and others. Jan. 1903. 2s. (Longmans, Green & Co.)

The Logarithm as a Direct Function. J. W. BRADSHAW. *On Positive Quadratic Forms.* P. SAUREL. *Multiple Points on Lissajou's Curves.* HOOK. *A Special Quadri-Quadric Transformation of Real Points in a Plane.* C. C. ENGBERG.

Bulletin of the American Mathematical Society. Edited by F. MORLEY and others. Feb. 1903. (Macmillan Co.)

On the Transformation of the Boundary in the Case of Conformal Mapping. W. F. OSGOOD. *On the Quintic Scroll having Three Double Conics.* V. SNYDER. *Surfaces referred to their Lines of Length Zero.* L. P. EISENHART.

The American Journal of Mathematics. Edited by F. MORLEY and others. Jan. 1903. (Kegan Paul.)

The Parametric Representation of the Tetrahedroid Surface. D. N. LEHMER. *On Ternary Monomial Substitution-Groups of Finite Order with Determinant ± 1 .* E. B. SKINNER. *The Forms of Unicursal Sextic Scrolls, etc.* V. SNYDER. *Note on Symmetric Functions.* E. D. ROE.

The Schoolmaster's Year Book and Directory for 1903. First annual issue. 5s. net. 1903. (Swan, Sonnenschein.)

Short Cuts and By-ways in Arithmetic. By C. BURCH. Pp. x, 107. 2s. 1903. (Blackie.)

A Complete Short Course of Arithmetic, mainly practical. By A. E. LAYNG. Pp. viii, 220. 1s. 6d. 1903. (Blackie.)

Quaternions and Projective Geometry. By C. J. JOLY. (Proceedings of Royal Society. Vol. 71.)

An Algebra for High Schools and Academies. By L. P. JOCELYN. Pp. 445. n.d. \$1.25. (The American Book Co. New York.)

The Arithmetic of Commerce and Trade. By S. JACKSON. Pp. 164. n.p., n.d. (Allman.)

ERRATA.

p. 248 line 12 up.	For	"from"	read	"form."
p. 248 " 10	"	" $\frac{1}{2}$ "	"	" $\frac{1}{3}$."
p. 248 " 12	Omit	"again."		
p. 249 " 29	For	"Goldback"	read	"Goldbach."
p. 249 " 30	"	"integer"	"	"even number."

BLACKIE & SON'S LIST,

WRITE FOR COMPLETE EDUCATIONAL CATALOGUE.

NEW GEOMETRY. For Beginners. Theoretical and Practical.

By RAWDON ROBERTS, B.Sc. A highly practical text-book of Geometry on modern lines, providing beginners with a mental training similar to that afforded by a study of the earlier books of Euclid's Elements. It deals with the properties of Rectilineal Figures, Similar Figures, and the Circle, and from the very beginning is replete with exercises of both a practical and theoretical nature. With Test Papers, Hints for the Solution of the Questions, and a Section on Curve Tracing. Crown 8vo, cloth, 1s. 6d.

A COMPLETE SHORT COURSE OF ARITHMETIC.

By A. E. LAYNG, M.A. A cheap, concise, highly practical, and clearly printed Arithmetic, covering the whole ground of the ordinary school course, and provided in addition with separate sections on mental and other short and approximate methods of calculation. With many fully-worked out examples and a large number of new and carefully assorted exercises. With Answers, 1s. 6d. Without Answers, 1s.

SHORT CUTS AND BY-WAYS IN ARITHMETIC.

By CECIL BURCH, late of the Statistical Department of the Council of Foreign Bondholders, London. With full explanations of all the principles involved, and numerous examples worked out at length. A large number of exercises for practice, with Answers, is also appended, together with Tables, etc., and a chapter on Magic Squares. Published price 2s. Copy post free, 1s. 9d.

LAYNG'S ARITHMETIC. By A. E. LAYNG, M.A. Crown 8vo, cloth, 4s. 6d. PART I. To Decimals and the Unitary Method. With or without Answers, 2s. 6d. PART II. Unitary Method to Scales of Notation. With or without Answers, 2s. 6d.

"A useful *résumé* is given of the first four rules: the other branches are treated in a very full manner.

A special excellence of the book is the large number of problems worked out as model solutions. The exercises . . . are numerous, well graduated, and very clearly printed. We heartily recommend the book."—*JOURNAL OF EDUCATION*:

"The examples are very well chosen and numerous. The book has the further merit of being cheap and at the same time well printed, and this, combined with its other good qualities, should ensure its success."—*GUARDIAN*.

LAYNG'S EXERCISES IN ARITHMETIC. (The Exercises of the above, published separately.) 2s. 6d. With Answers, 3s. PART I. (5000 Exercises), without Answers, 1s. With Answers, 1s. 6d. PART II. (3500 Exercises), without Answers, 1s. 6d. With Answers, 2s.

A NEW SEQUEL TO EUCLID.

By Professor W. J. DILWORTH, M.A. 2s. 6d. Also in Two Parts. Part I., 1s.; Part II., 2s.

"This little manual surpasses the generality of its many competitors in judicious arrangement of matter and clearness of demonstrations. Modern technical terms are defined, over fifty easy and useful propositions not found in Euclid are proved, and a sufficient number of others are given unproved as exercises for the ingenuity of the student."—*ATHENÆUM*.

EUCLID'S ELEMENTS OF GEOMETRY. With Notes, Examples, and Exercises. Edited by A. E. LAYNG, M.A. Books I. to VI., and XI., with Appendix; and a wide selection of Examination Papers. 3s. 6d.

Book I., 1s.; II., 6d.; III., 1s.; IV., 6d. V. and VI. together, 1s.; XI., 1s. 6d. Books I.-II., 1s. 3d.; I.-III., 2s.; I.-IV., 2s. 6d. KEY to Book I., 2s. 6d.; to complete Euclid, 5s.

The system of arrangement allows enunciation, figure, and proof to be all in view together. Notes and Exercises are directly appended to the propositions to which they refer.

"The demonstrations are neatly and concisely put, with a line for each step in the argument. . . . From its excellent arrangement, and the beauty of the type and diagrams, we should think that Mr. Layng's *Euclid* will become a favourite with many teachers. There is an excellent series of subsidiary propositions. . . . Mr. Layng's proof of the 36th (prop. of Book III.) is the nearest we have seen recently."—*SATURDAY REVIEW*.

EASY MATHEMATICAL PROBLEM PAPERS.

By CHARLES DAVIDSON, Sc.D. Containing numerous problems in Algebra to the Binomial Theorem, Euclid, Books I.-VI. and XI., and Trigonometry to the Solution of Triangles and Geometrical Applications. With or without Answers and Hints for Solution. 2s. 6d.

BLACKIE & SON, LTD., 50 Old Bailey, London, E.C.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament therunto."--BACON.

President :

Professor A. R. FORSYTH, LL.D., Sc.D., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.

R. LEVETT, M.A.

Prof. W. H. H. HUDSON, M.A.

Professor A. LODGE, M.A.

Professor G. M. MINCHIN, F.R.S.

Treasurer :

F. W. HILL, M.A., City of London School, London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.

H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

S. BARNARD, M.A.

F. S. MACAULAY, M.A., D.Sc.

R. F. DAVIS, M.A.

W. N. ROSEVEARE, M.A.

J. M. DYER, M.A.

A. W. SIDMONS, M.A.

C. GODFREY, M.A.

C. O. TUCKEY, B.A.

G. HEPPEL, M.A.

A. E. WESTERN, M.A.

C. E. WILLIAMS, M.A.

THE MATHEMATICAL ASSOCIATION, formerly known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all masters and mistresses, who are interested in promoting good methods of mathematical teaching. Such an Association should be a recognized authority in its own department, and should exert an important influence on methods of examination.

General Meetings of the Association are held in London once a term, and in other places if desired. At these Meetings papers on elementary mathematics are read, and any member is at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

"The *Mathematical Gazette*" is the organ of the Association. It contains—

(1) ARTICLES, on subjects within the scope of elementary mathematics.

(2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books.

(3) REVIEWS, at present the most striking feature of the *Gazette*, and written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell rather on the general development of the subject, than upon the part played therein by the book under notice.

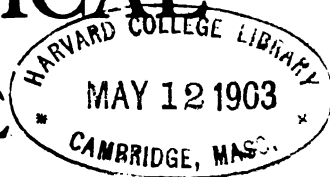
(4) PROBLEMS AND SOLUTIONS, generally selected to show the trend of investigation at the universities, so far as is shown in the most recent scholarship papers. Questions of special interest or novelty also find a place in this section.

(5) SHORT NOTICES, of books not specially dealt with in the REVIEWS.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "The *Mathematical Gazette*."

June 7th 1903

THE MATHEMATICAL GAZETTE



EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.
E. T. WHITTAKER, M.A.
W. E. HARTLEY, B.A.

LONDON
GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. II., No. 39.	MAY, 1903.	1s. 6d. Net.
-------------------	------------	--------------

The Mathematical Association.

NOTICE.

A MEETING of the MATHEMATICAL ASSOCIATION will be held at KING'S COLLEGE, STRAND, LONDON, W.C., at 3 p.m. on *Saturday, May 23rd, 1903.* The following Papers will be read:—

1. "The Slide Rule and its use in teaching Logarithms," by C. S. JACKSON, M.A.
2. "A Geometrical Note," by R. F. DAVIS, M.A.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, 21 Argyll Place, Aberdeen.

All other correspondence concerning the Gazette should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

Correspondence relative to the Mathematical Association should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, St. Paul's School, London, W.; H. D. ELLIS, 12 Gloucester Terrace, Hyde Park, W.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied, except on payment of 1s. per copy.

CONTENTS.

	PAGE
ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE. Prof. A. LODGE, ...	277
AN ELEMENTARY INTRODUCTION TO THE INFINITESIMAL GEOMETRY OF SURFACES. R. W. H. T. HUDSON, ...	279
PROOFS THAT THE ARITHMETIC MEAN IS GREATER THAN THE GEOMETRIC MEAN. R. F. MUIRHEAD, ...	283
MATHEMATICAL NOTES. Rev. W. H. LAVERTY; C. E. M'VICKER; Prof. G. H. BRYAN, F.R.S., ...	287
REVIEWS. Prof. G. B. MATHEWS, F.R.S.; E. M. LANGLEY, ...	290
PROBLEMS, ...	295
SOLUTIONS, ...	296
SALE AND EXCHANGE, ...	299
BOOKS, ETC., RECEIVED, ...	299
ERRATA, ...	300

Forthcoming Articles and Reviews by:—

Prof. A. C. DIXON; Prof. T. J. I'a BROMWICH; Prof. A. LODGE; Prof. G. B. MATHEWS; W. G. BELL, M.A.; J. H. GRACE, M.A.; R. W. H. T. HUDSON, M.A.; E. M. LANGLEY, M.A.; W. N. ROSEVEARE, M.A.; W. P. WORKMAN, M.A.; H. W. RICHMOND, M.A.; A. N. WHITEHEAD, M.A.; E. W. BARNES, M.A.; W. F. BEARD, M.A.; F. S. MACAULAY, D.S.C. and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. *No. 8 is out of print.*
A few numbers may be obtained from the Editor at the following prices, post free:

No. 7, 5s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 15s.

These will soon be out of print and very scarce.

NOW READY.

SOLID GEOMETRY

By CHARLES GODFREY, M.A.

Translated and adapted from the German of DR. FRANZ HOEVAR.
Small Crown 8vo, with 50 Illustrations. *PRICE 1s. 6d.*

NOW READY.

BEGINNERS' ALGEBRA

By M. S. DAVID, B.A.

Small Crown 8vo. Illustrated. *PRICE 2s. 6d.*

A. & C. BLACK, Soho Square, LONDON, W.

THE MATHEMATICAL



EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

MAY, 1903.

No. 39.

ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE.

IN plane coordinate geometry, with a system of rectangular axes, I wish to press the claim of points whose coordinates are given by complex numbers to be represented on the plane in just the same way as all other points in the plane.

It will doubtless be conceded that when we plot a point from its coordinates (x, y) we are in reality plotting its position by means of the complex quantity $x+iy$. This we habitually do when x and y are simple, or 'real' lengths, and my contention is that we ought to extend the privilege to *all* points whatever.

If we allow ourselves to do this when x and y are complex numbers, we can always represent 'imaginary' points by real ones. Such plotted points may be called the representatives of the imaginary points; or, for brevity, they may be considered as being actually the imaginary points, which will thus be real in position, and imaginary only in the sense that they are not visibly situated in the locus whose equation they satisfy by virtue of their particular complex coordinates.

Thus, let $(m+ip, n+iq)$ be the complex coordinates of a point. We plot it as $(m+ip)+i(n+iq)$, i.e. $(m-q)+i(n+p)$, its simple coordinates being $(m-q, n+p)$.

The conjugate complex, viz. $(m-ip, n-iq)$ would similarly have the simple coordinates $(m+q, n-p)$.

In this way the pair of complexes $(m \pm ip, n \pm iq)$, and their pair of simple correlatives $(m \mp q, n \pm p)$, both uniquely represent the positions of the two points given by the complex relations $(m+in) \pm i(p+iq)$.

There is a perfect 2 to 2 correspondence between the first and second pair of points.

But there is a distinction.

The points $(m+ip, n+iq)$ and $(m-ip, n-iq)$ satisfy the equation

$$\frac{x-m}{p} = \frac{y-n}{q},$$

while the points $(m-q, n+p)$ and $(m+q, n-p)$ satisfy the equation

$$p(x-m)+q(y-n)=0;$$

i.e. the first pair of points are imaginary points on one line, and the second pair are real points on a line at right angles to the first. Both lines pass

through (m, n) . The distance of the first pair of points from (m, n) is $i\sqrt{(p^2+q^2)}$, and the distance of the second pair from (m, n) is $\sqrt{(p^2+q^2)}$.

Hence, two conjugate imaginary points on a line are represented by a pair of real points which are each other's *images* in the line, at a distance from it equal to the modulus of the purely imaginary part of the complex coordinates, their mid-point having as coordinates the purely real parts of the same coordinates.

Now the imaginary points on a line are doubly infinite in number, covering the whole plane. The above investigation shows that they do not lie haphazard on it, but that conjugate points are 'image' points in the line. As a corollary it follows that the plane can be stratified, as it were, into a series of loci consisting of conjugate imaginary points on the line, which will have the line as an axis of symmetry and will cross it at *right angles*, if at all, unless they have some singularity, such as a cusp or double point, where they cross it. And, conversely, any locus which has the line as an axis of symmetry consists of a series of conjugate imaginary points on the line.

Now take the case of a pair of imaginary points on a circle. This we can best do by considering the intersections of a non-cutting straight line with the circle.

Evidently the circle is only one of a coaxial system, cutting the line in the same two points. Hence the limiting points of the system are the representatives of the imaginary points of intersection. Consequently the imaginary points of intersection lie on the radius which cuts the line orthogonally, at distances (r_1, r_2) from the centre given by the equations $r_1 r_2 = a^2$, and $r_1 + r_2 = 2b$, where a is the radius of the circle, and b the distance of the line from the centre of the circle.

By moving the line in any systematic manner, we can obtain a *locus* of imaginary points which will (except for singularities) cross the circle orthogonally if at all. Thus, if we move the line parallel to itself, the locus is a diameter of the circle perpendicular to the series of moving lines; if we rotate it round any point on it, we obtain an orthogonal circle, and so on. All loci so obtained will have their parts which are outside the circle the inverses of the parts which are inside the circle.

Now take the case of a conic. We will treat this analytically by finding the points where the ellipse

$$b^2 x^2 + a^2 y^2 = a^2 b^2 \dots\dots\dots (1)$$

is cut by any line $bx \cos \phi + ay \sin \phi = kab, \dots\dots\dots (2)$

where k is greater than 1.

We shall find that if we move the line parallel to itself, the intersection points lie on a confocal conic which cuts the given conic orthogonally at the point of contact of the parallel tangent

$$bx \cos \phi + ay \sin \phi = ab. \dots\dots\dots (3)$$

Consequently the whole plane can be stratified, as it were, into a series of confocal conics, each confocal being the locus of the pairs of imaginary points in which a series of parallel external lines cut the given conic.

To prove this, solve (1) and (2) simultaneously, remembering that k is greater than unity.

We find, for the points of intersection,

$$\begin{aligned} x &= ka \cos \phi \pm ia \sin \phi \sqrt{(k^2 - 1)} \\ y &= kb \sin \phi \mp ib \cos \phi \sqrt{(k^2 - 1)} \end{aligned} \dots\dots\dots (4)$$

their position being represented on the diagram by $x+iy$ in each case, the values of their corresponding simple coordinates being

$$\begin{aligned} x &= ka \cos \phi \pm b \cos \phi \sqrt{(k^2 - 1)} \\ y &= kb \sin \phi \pm a \sin \phi \sqrt{(k^2 - 1)} \end{aligned} \dots\dots\dots (5)$$

If we now eliminate k , we find for the locus of the points of intersection of the series of parallel chords with the ellipse, the curve

$$x^2 \sec^2 \phi - y^2 \operatorname{cosec}^2 \phi = a^2 - b^2. \dots\dots\dots(6)$$

This is a confocal hyperbola, passing through the point $(a \cos \phi, b \sin \phi)$ which is the point of contact of (3) with (1), and which we may call the point P .

The asymptotes to this hyperbola make the angle ϕ with the axis of x , so that they pass through the points where the auxiliary circle of the ellipse is cut by the ordinate at P .

The mid-points of the pairs of conjugates lie on the diameter

$$bx \sin \phi = ay \cos \phi, \dots\dots\dots(7)$$

i.e. on the diameter passing through P .

The line through the mid-point M of any one of such pairs drawn perpendicular to the system of parallel chords will cut the hyperbola in the required pair of points. The equation of this line is

$$ax \sec \phi - by \operatorname{cosec} \phi = k(a^2 - b^2). \dots\dots\dots(8)$$

The pole of (8) with regard to the hyperbola is

$$\left(\frac{a \cos \phi}{k}, \frac{b \sin \phi}{k} \right),$$

which is also the pole of the line (2) with regard to the ellipse. We are thus led to the very interesting theorem that the tangents to the hyperbola drawn from the representative points (5) where the given chord (2) cuts the given ellipse intersect in the pole of the chord with respect to the ellipse.

Consequently not only do real points on the hyperbola represent imaginary points on the ellipse, but real tangents to the hyperbola represent imaginary tangents to the ellipse.

There are other interesting things connected with pairs of conjugate imaginary points on a conic, but the above are sufficient to show that the recognition of the right of imaginary points to be elevated to the dignity of plottable points on the plane will be rewarded by interesting developments. The main object of this paper is to plead that right.

Some of the developments are likely to be important, as evidently the matter is closely connected with the theory of orthogonal curves.

The connection of the above with the Argand diagram is too obvious to need pointing out. It shows that the method of plotting by rectangular coordinates is an extension of the Argand diagram to two variables. It has many interesting applications in the theory of equations.

A. LODGE.

COOPER'S HILL.

AN ELEMENTARY INTRODUCTION TO THE INFINITESIMAL GEOMETRY OF SURFACES.

The purpose of the present note is to show how the student who is acquainted with the methods only of using and transforming rectangular axes is already in a position to determine the geodesic curvature and torsion of any curve traced on a surface, and in particular to prove Gauss' formula for the product of the principal curvatures in terms of the geodesic curvatures of the lines of curvature.

If the axes of x and y are turned in their plane about the origin through an angle θ , the formulae of transformation of coordinates are

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta, \end{aligned}$$

leading to

$$ax^2 + 2hxy + by^2 = a'x'^2 + 2h'x'y' + b'y'^2,$$

where

$$a' = a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta,$$

$$h' = (b - a) \cos \theta \sin \theta + h(\cos^2 \theta - \sin^2 \theta),$$

$$b' = a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta,$$

so the term in $x'y'$ disappears when the angle of rotation is given by

$$\tan 2\theta = 2h/(a - b).$$

When θ is so small that θ^2 may be neglected, these formulae become

$$x = x' - y'\theta,$$

$$y = x'\theta + y'.$$

The change of coordinates due to rotating the frame of reference through an angle θ_1 about the axis of x , and then through an angle θ_2 about the axis of y' , when squares of the angles are neglected, is found by combining the transformations

$$\left. \begin{aligned} x &= x' \\ y &= y' - x'\theta_1 \\ z &= z' + y'\theta_1 \end{aligned} \right\} \text{ and } \left. \begin{aligned} y' &= y'' \\ x' &= x'' - y''\theta_2 \\ z' &= z'' - x''\theta_2 + y''\theta_1 \end{aligned} \right\},$$

which lead to

$$z = z'' - x''\theta_2 + y''\theta_1,$$

showing that the order of the rotations is immaterial. Hence the plane $z'' = 0$ is, when referred to the original axes,

$$z + x\theta_2 - y\theta_1 = 0.$$

We define the *normal curvature* of a curve traced on a surface as the component rate at which the tangent plane turns about the line in itself perpendicular to the tangent line to the curve, and the *geodesic torsion* as the component rate at which the tangent plane turns about the tangent line, the point of contact being supposed to describe the curve at unit rate.

Let the equation of a surface be expressed in the form

$$z = u_2 + u_3 + u_4 + \dots,$$

where

$$2u_2 = ax^2 + 2hxy + by^2,$$

and $u_3, u_4, \dots$ are homogeneous polynomials in x, y of degrees indicated by their suffixes, so that the origin is on the surface, and the axis of z is normal to it.

Change the origin to $(\delta s_1, 0, 0)$ and neglect the square of δs_1 ; the new equation is

$$z - \frac{\partial u_2}{\partial x} \delta s_1 = u_2 + \frac{\partial u_3}{\partial x} \delta s_1 + u_3 + \frac{\partial u_4}{\partial x} \delta s_1 + \dots,$$

showing that the tangent plane at the new origin is

$$z - (ax + hy) \delta s_1 = 0.$$

On comparing this with

$$z + x\theta_2 - y\theta_1 = 0,$$

we see that the tangent plane has undergone rotations

$$\theta_1 = h \delta s_1,$$

$$\theta_2 = -a \delta s_1,$$

so, without paying attention to sign, we may say that a is the normal curvature and h the geodesic torsion of the section by $y = 0$. Similarly, for a

displacement δs_2 of the point of contact along the section by $x=0$, the tangent plane undergoes rotations $b\delta s_2$ and $-h\delta s_2$. We infer that any two curves on a surface which cut at right angles have the same (numerical) geodesic torsion.

Since by turning the axes of x and y it is possible to get rid of the term in xy , we infer that there are two curves through any point of the surface which have zero geodesic torsion; this may be taken to be the definition of *lines of curvature*, and when the term in xy is absent, the coefficients of x^2 and y^2 are called the *principal curvatures*.

The familiar forms for normal curvature and geodesic torsion are found simply by turning the axes of x and y through an angle θ . If then the equation

$$2z = \kappa_1 x^2 + \kappa_2 y^2 + u_3 + \dots$$

becomes

$$2z = ax^2 + 2hxy + by^2 + \dots,$$

we have

$$a = \kappa_1 \cos^2 \theta + \kappa_2 \sin^2 \theta,$$

giving Euler's formula for normal curvature, and

$$h = (\kappa_2 - \kappa_1) \cos \theta \sin \theta,$$

giving the torsion of any geodesic.

In future we shall take $a = \kappa_1$, $h = 0$, $b = \kappa_2$, so that the standard form of equation is

$$2z = \kappa_1 x^2 + \kappa_2 y^2 + u_3 + \dots,$$

where

$$3u_3 \equiv Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3,$$

$$12u_4 \equiv ax^4 + 4\beta x^3y + 6\gamma x^2y^2 + 4\delta xy^3 + ey^4.$$

In order to find the rates of change of the principal curvatures we transfer the origin to $(\delta s_1, 0, 0)$, and then turn the axes until the equation is again reduced to standard form; throughout the work squares of displacements are neglected. In the equation

$$2(z - \kappa_1 x \delta s_1) = \kappa_1 x^2 + \kappa_2 y^2 + \frac{\partial u_3}{\partial x} \delta s_1 + \dots$$

we must replace

$$z \text{ by } z + \kappa_1 x \delta s_1$$

and

$$x \text{ by } x - \kappa_1 z \delta s_1,$$

$$= x - \frac{1}{2} \kappa_1 (\kappa_1 x^2 + \kappa_2 y^2) \delta s_1 + \dots$$

This does not affect the quadratic terms, which are

$$(\kappa_1 + A \delta s_1) x^2 + 2B \delta s_1 xy + (\kappa_2 + C \delta s_1) y^2.$$

To remove the middle term turn the axes through the small angle $B\delta s_1/(\kappa_1 - \kappa_2)$, thereby leaving the coefficients of x^2 and y^2 unaltered. The equation is now of the standard form and we may indicate the changes which the coefficients have undergone by the symbol δ . Thus

$$\delta \kappa_1 = A \delta s_1, \quad \delta \kappa_2 = C \delta s_1,$$

that is to say, the coefficients A and C of the original equation are the rates of change of the principal curvatures along the first line of curvature. Similarly we may write $B = \delta \kappa_1 / \delta s_2$, $D = \delta \kappa_2 / \delta s_2$. It must carefully be noticed that the symbol δ is not the same as the symbol of differentiation, for although δs_1 , δs_2 are elements of length, they are not perfect differentials of functions of the coordinates, and so the integrated forms s_1 , s_2 have no definite meaning; much less therefore can κ_1 , κ_2 be regarded as functions of s_1 and s_2 .

We have seen that when the origin moves along a line of curvature and the axis of x is tangent to it, the axis of z being normal to the surface, then

the frame turns about the z -axis at a rate $B/(\kappa_1 - \kappa_2)$, $=g_1$ say. Then g_1 is called the *geodesic curvature* of the line of curvature, and is positive when the turning is from x towards y . Similarly $C/(\kappa_2 - \kappa_1)$, $=g_2$ say, is the geodesic curvature of the other line of curvature and is positive when the turning is from y towards x . In order to find Gauss' expression for $\kappa_1\kappa_2$ in terms of g_1, g_2 and their rates of change, we require to know $\delta C/\delta s_1$. In order to bring back the equation

$$2(z - \kappa_1 x \delta s_1) = \kappa_1 x^2 + \kappa_2 y^2 + \frac{\partial u_3}{\partial x} \delta s_1 + u_3 + \frac{\partial u_4}{\partial x} \delta s_1 + \dots$$

to standard form we have, as was shown above, to replace

$$x \text{ by } x - \frac{1}{2}\kappa_1(\kappa_1 x^2 + \kappa_2 y^2)\delta s_1 - y g_1 \delta s_1,$$

$$y \text{ by } y + x g_1 \delta s_1,$$

$$z \text{ by } z + \kappa_1 x \delta s_1,$$

and then the coefficient of xy^2 arises from the terms

$$\kappa_1 x^2 + Bx^2y + (C + \gamma \delta s_1)xy^2 + \frac{1}{2}Dy^3,$$

and is

$$-\kappa_1^2 \kappa_2 \delta s_1 - 2Bg_1 \delta s_1 + C + \gamma \delta s_1 + Dg_1 \delta s_1.$$

Accordingly we write

$$\delta C/\delta s_1 = \gamma - \kappa_1^2 \kappa_2 + (BD - 2B^2)/(\kappa_1 - \kappa_2).$$

Similarly

$$\delta B/\delta s_2 = \gamma - \kappa_1 \kappa_2^2 + (CA - 2C^2)/(\kappa_2 - \kappa_1),$$

whence

$$\begin{aligned} \delta g_1/\delta s_2 + \delta g_2/\delta s_1 &= (\delta B/\delta s_2 - \delta C/\delta s_1)/(\kappa_1 - \kappa_2) \\ &\quad - \{B(B - D) + C(C - A)\}/(\kappa_1 - \kappa_2)^2 \\ &= \kappa_1 \kappa_2 + (B^2 + C^2)/(\kappa_1 - \kappa_2)^2; \\ \therefore \kappa_1 \kappa_2 &= \delta g_1/\delta s_2 + \delta g_2/\delta s_1 - g_1^2 - g_2^2, \end{aligned}$$

which is the formula required.

Since squares of small quantities are being neglected, the displacements $\delta s_1, \delta s_2$ may be superposed by addition. Hence when the origin is moved to $(\delta s_1, \delta s_2, 0)$ the frame must undergo a rotation

$$g_1 \delta s_1 - g_2 \delta s_2$$

in order that the axes may be again tangents to lines of curvature. If then a curve makes a variable angle θ with the first line of curvature through any point of it, a frame having one axis always tangent to it and another normal to the surface must turn about the latter through an angle

$$g_1 \delta s_1 - g_2 \delta s_2 + \delta \theta,$$

from which we deduce that the rate of turning, which by definition is the geodesic curvature, is

$$g_1 \cos \theta - g_2 \sin \theta + d\theta/ds.$$

Having found the normal curvature, geodesic curvature and geodesic torsion of any curve on a surface, we are in a position to determine its curvature, torsion, and all other intrinsic properties.

Interesting examples of these methods are afforded by asymptotic curves and parabolic curves, but it is unnecessary here to pursue the details.

R. W. H. T. HUDSON.

LIVERPOOL, November, 1902.

PROOFS THAT THE ARITHMETIC MEAN IS GREATER THAN THE GEOMETRIC MEAN.

What follows is a condensed statement and classification of various proofs of the above well-known theorem.

For two positive quantities, a and b , the proof may be put as follows :

$$\frac{a+b}{2} - \sqrt{ab} = \frac{(\sqrt{a} - \sqrt{b})^2}{2}, \text{ which is } > 0.$$

$$(1) \quad \therefore \frac{a+b}{2} > \sqrt{ab}.$$

Otherwise, let the two quantities be denoted by $x+y$ and $x-y$, then their arithmetic mean x is $>$ their geometric mean $\sqrt{x^2 - y^2}$.

Here we assume for the sake of simplicity, as we shall do throughout, that the given quantities are not all equal ; otherwise the symbol $\leq$ would have to be used instead of $>$. We shall use A and G to denote the means.

To extend the theorem to n positive quantities $a, b, c, \dots$, several methods are available, which may be classified as follows :

A. By substituting for the greatest and least another equal pair, which (α) leaves the A.M. unaltered and increases the G.M., or (β) leaves the G.M. unaltered and diminishes the A.M., but in both cases makes the A.M. and the G.M. continually approach equality.

B. By substituting for a and b , the greatest and least of the set, a pair of quantities having (α) the same G.M., or (β) the same A.M. as a and b , one of the pair being the mean of all ; and so by a *finite* number of steps completing the proof.

C. By proving the theorem for $n=2^m$, and extending it to the general case by introducing $2^m - n$ equal quantities (α) each equal to G , or (β) each equal to A .

D. By mathematical induction, making the case for n quantities depend on that for $n-1$. Here also are two varieties, (α) and (β), in which A and G exchange rôles, and a third, (γ), depending on the inequality

$$\left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^{n-1}.$$

E. By showing that A and G are the extremes of a set of means of cognate type which are in descending order of magnitude.

A α . Substitute for a and b , the greatest and least of the n given quantities, a pair of quantities each $= (a+b)/2$. We have thus a new set of n quantities whose sum is the same as that of the given set, but whose continued product is greater, since by (1),

$$\left(\frac{a+b}{2}\right)^2 > ab \text{ and } \therefore \left(\frac{a+b}{2}\right)^2 \cdot cde \dots > abcde \dots$$

By taking a sufficient number of such steps we ultimately get a set having the same sum as $a, b, c, \dots$ but a greater product, the members of which are as nearly equal to one another, and therefore to their G.M., as we please. At the same time this G.M. approaches as nearly as we please to the A.M. which is the same for all the sets. Since the G.M. of the last set is $>$ that of the given set by a finite amount, it follows that the G.M. of the latter is less than its A.M.

A β . Substitute for a and b their G.M. $\sqrt{ab}$ taken twice, and we get a new set of n quantities having the same continued product as that of $a, b, c, \dots$, but a smaller sum, since

$$2\sqrt{ab} - a - b = -(\sqrt{a} - \sqrt{b})^2, \text{ which is } < 0.$$

Taking a sufficient number of such steps, we ultimately get a set of numbers as nearly equal to one another, to their A.M., and to their G.M., as we please, and having the same product as $a, b, c, \dots$, but a smaller sum. Hence we deduce that the A.M. of the original set is greater than their G.M.

In order to make the proofs $A\alpha$ and $A\beta$ complete, it would be necessary to show that by taking a sufficient number of steps in each case, we can make the numbers of the resulting set approach mutual equality as nearly as we please.

Ba. Assuming for the present that both the arithmetic and the geometric means of n quantities lie between b and a , the least and the greatest of them, G , the geometric mean, will lie between a and b . Let then $k = \frac{ab}{G}$, so that

$$a+b-(G+k) = a+b-G-\frac{ab}{G} = \frac{1}{G}(a-G)(G-b), \text{ which is } > 0;$$

$$\therefore a+b > G+k.$$

Thus we have a new set of quantities having the same continued product as the given set but a smaller sum, and including at least one quantity equal to G . By a series of similar steps, not more than $n-1$ in number, we arrive at n quantities G , having a smaller sum than the given set, whose A.M. is therefore $> G$.

B β . Next let A be the arithmetic mean of the given numbers, and let $k = a+b-A$, where, as before, a and b stand for the greatest and least of the set considered. Then A lies between a and b . Again,

$$Ak-ab = A(a+b-A)-ab = (a-A)(A-b), \text{ which is } > 0.$$

Thus taking A and k for a and b , we have a new set of quantities having the same sum, but a greater continued product, and including at least one quantity equal to A . By a number of similar steps, $n-1$ at most, we arrive at a set of n quantities, each equal to A , which have a greater continued product than $a, b, c, \dots$ and whose G.M. is therefore $> G$. Hence $A > G$.

To prove the assumptions that A and G lie between a and b , we have only to note that

$$(2) \quad na - nA = (a-a) + (a-b) + (a-c) + \dots, \text{ which is } > 0, \text{ so that } a > A.$$

$$(3) \quad nA - nb = (a-b) + (b-b) + (c-b) + \dots, \text{ which is } > 0, \text{ so that } A > b.$$

$$(4) \quad a^n \div G^n = a^n \div (abc \dots) = \frac{a}{a} \cdot \frac{a}{b} \cdot \frac{a}{c} \dots, \text{ which is } > 1, \text{ so that } a > G.$$

$$(5) \quad G^n \div b^n = (abc \dots) \div b^n = \frac{a}{b} \cdot \frac{b}{b} \cdot \frac{c}{b} \dots, \text{ which is } > 1, \text{ so that } G > b.$$

C. To prove the theorem when $n = 2^m$, observe that

$$\begin{aligned} & a+b+c+\dots-n(a \cdot b \cdot c \dots)^{\frac{1}{n}} \\ &= (a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + (c^{\frac{1}{2}} - d^{\frac{1}{2}})^2 + \dots \\ &+ 2\{(a^{\frac{1}{4}}b^{\frac{1}{4}} - c^{\frac{1}{4}}d^{\frac{1}{4}})^2 + (e^{\frac{1}{4}}f^{\frac{1}{4}} - g^{\frac{1}{4}}h^{\frac{1}{4}})^2 + \dots\} \\ &+ 2^2\{(a^{\frac{1}{8}}b^{\frac{1}{8}}c^{\frac{1}{8}}d^{\frac{1}{8}} - e^{\frac{1}{8}}f^{\frac{1}{8}}g^{\frac{1}{8}}h^{\frac{1}{8}})^2 + \dots\} \\ &+ \dots + \frac{n}{2}\{(ab \dots)^{\frac{1}{n}} - (uv \dots)^{\frac{1}{n}}\}, \text{ which is } > 0. \end{aligned}$$

Hence

$$(a+b+c \dots)/n > (abc \dots)^{\frac{1}{n}}.$$

Q.E.D.

Ca. Next suppose n not a power of 2, and let $n+p=2^m$.

$$\begin{aligned} a+b+c\ldots+pG &> (n+p)(abc\ldots G^p)^{\frac{1}{n+p}} \\ &> (n+p)(G^n G^p)^{\frac{1}{n+p}} \\ &> (n+p)G. \end{aligned}$$

$$\therefore a+b+c\ldots > nG,$$

whence

$$A > G.$$

Cβ. Again, $a+b+c+\ldots+pA > (n+p)(abc\ldots A^p)^{\frac{1}{n+p}}$.

$$\therefore nA+pA > (n+p)(G^n A^p)^{\frac{1}{n+p}}.$$

$$\therefore A^{n+p} > G^n A^p.$$

$$\therefore A > G.$$

Da. Arrange $a_1 a_2 \ldots$ in ascending order, so that $a_n > G_{n-1}$ (see (4)), and let

$$G_{n-1} \equiv (a_1 a_2 \ldots a_{n-1})^{\frac{1}{n-1}} \quad \text{and} \quad G_n \equiv (a_1 a_2 \ldots a_n)^{\frac{1}{n}}.$$

Assuming

$$a_1 + a_2 + \ldots + a_{n-1} > (n-1)G_{n-1}.$$

$$\therefore a_1 + a_2 + \ldots + a_{n-1} + a_n > (n-1)G_{n-1} + a_n$$

$$\therefore \frac{a_1 + a_2 + \ldots + a_n}{n} > G_{n-1} + \frac{a_n - G_{n-1}}{n}.$$

$$\therefore A_n^n > \left(G_{n-1} + \frac{a_n - G_{n-1}}{n} \right)^n$$

$$> G_{n-1}^n + n \cdot G_{n-1}^{n-1} \times \frac{a_n - G_{n-1}}{n}$$

by the Binomial Theorem, since $a_n - G_{n-1}$ is positive.

$$\therefore A_n^n > G_{n-1}^{n-1} a_n$$

$$> a_1 a_2 \ldots a_{n-1} a_n$$

$$\therefore A_n > G_n.$$

Hence, starting from (1), by mathematical induction, $A > G$.

Dβ. Assume $A_{n-1} > G_{n-1}$. By (2) $a_n > A_{n-1}$.

$$\therefore A_n = \frac{n-1 A_{n-1} + a_n}{n} = A_{n-1} + \frac{a_n - A_{n-1}}{n}.$$

$$\therefore A_n^n > A_{n-1}^n + n A_{n-1}^{n-1} \times \frac{a_n - A_{n-1}}{n}$$

$$> a_n A_{n-1}^{n-1} > a_n G_{n-1}^{n-1} > G_n^n.$$

$$\therefore A_n > G_n.$$

As before, by mathematical induction, $A > G$.

Dγ. Thacker's proof depends on this lemma: if x is positive, and n a positive integer,

$$\left(1 + \frac{x}{n}\right)^n > \left(1 + \frac{x}{n-1}\right)^{n-1};$$

which is proved by expanding both sides by the Binomial Theorem, and observing that the r th term of the former expansion is $>$ that of the latter, since

$$\frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \ldots \frac{n-r+2}{n} > \frac{n-1}{n-1} \cdot \frac{n-2}{n-1} \cdot \frac{n-3}{n-1} \ldots \frac{n-r+1}{n-1}.$$

Hence

$$\begin{aligned} A_n^n &= a_1^n \left(1 + \frac{a_1 + a_2 + \dots + a_n - na_1}{na_1} \right)^n \\ &> a_1^n \left(1 + \frac{a_2 + a_3 + \dots + a_n - (n-1)a_1}{(n-1)a_1} \right)^{n-1} \quad \text{by the lemma,} \\ &> a_1^n \left(\frac{a_2 + a_3 + \dots + a_n}{(n-1)a_1} \right)^{n-1} \\ &> a_1 \left(\frac{a_2 + a_3 + \dots + a_n}{n-1} \right)^{n-1}. \end{aligned}$$

Thus, assuming $A > G$ for $n-1$ quantities, we have

$$\begin{aligned} A_n^n &> a_1(a_2 a_3 \dots a_n) > G_n^n \\ \therefore A_n &> G_n \end{aligned}$$

Thus the theorem follows by mathematical induction.

E. Let $N_r \equiv (\text{sum of products of } a, b, c, \dots \text{ taken } r \text{ at a time}) \div (\text{number of such products})$, then it is proved in the Theory of Equations that

$$(6) \quad N_r^2 > N_{r-1}N_{r+1}. \quad \therefore \frac{N_r}{N_{r-1}} > \frac{N_{r+1}}{N_r}.$$

Hence

$$\begin{aligned} \frac{N_1}{1} &> \frac{N_2}{N_1} > \frac{N_3}{N_2} \dots > \frac{N_r}{N_{r-1}} > \frac{N_{r+1}}{N_r}. \\ \therefore \left(\frac{N_{r+1}}{N_r}\right)^r &< \frac{N_r}{N_{r-1}} \cdot \frac{N_{r-1}}{N_{r-2}} \dots \frac{N}{1} < N_r. \\ \therefore N_{r+1}^r &< N_r^{r+1}. \quad \therefore N_{r+1}^{\frac{1}{r+1}} < N_r^{\frac{1}{r}}. \end{aligned}$$

Hence

$$(7) \quad N_1 > N_2^{\frac{1}{2}} > N_3^{\frac{1}{2}} \dots > N_n^{\frac{1}{2}}.$$

But

$$N_1 \equiv A_n \quad \text{and} \quad N_n^{\frac{1}{2}} \equiv G_n.$$

Hence

$$A_n \geq G_n$$

The following formula, recently arrived at by the writer, expresses in a form which shows that it is necessarily positive, the excess of the arithmetic over the geometric mean of n quantities, written for convenience in the form $a^n, b^n, c^n, \dots$

$$\begin{aligned} \frac{1}{n} \Sigma a^n - abcd \dots &= \frac{1}{n(n-1)} \Sigma (a-b)(a^{n-1} - b^{n-1}) \\ &+ \frac{1!}{n(n-1)(n-2)} \Sigma (a-b)(a^{n-2} - b^{n-2})c \\ &+ \frac{2!}{n(n-1)(n-2)(n-3)} \Sigma (a-b)(a^{n-3} - b^{n-3})cd \\ &+ \dots \\ &+ \frac{(r-1)!(n-r-1)!}{n!} \Sigma (a-b)(a^{n-r} - b^{n-r})cd \dots \\ &+ \dots \\ &+ \frac{(n-2)!}{n!} \Sigma (a-b)^2 cd \dots \end{aligned}$$

where each term is of course of degree n in $a, b, c, d, \dots$

The following references, though not exhaustive, may be found useful :

A α is given by Todhunter and by Chrystal ; A α and A β are both given by C. Smith.

B α is due to G. H. Bryan (see the *Tutorial Algebra*), and B β to G. E. Crawford (*Edin. Math. Soc. Proceedings*, XVIII., p. 2).

C β is given in Todhunter's *Algebra*, and is due to Cauchy (*Analyse Algèbre*, p. 457) ; and C α is obviously suggested by it.

D β is given in H. Weber's *Algebra*, and D α is suggested by it. D γ is due to Thacker (*Camb. and Dub. Math. Journ.*, VI., p. 81). Another proof, of type D, but depending on a different fundamental inequality, is given by Chrystal (*Algebra*, XXIV., § 8).

E is given by Schlömilch in his *Zeitschrift* (III., p. 301), with a proof depending on Theory of Equations, and attributed to Fort. A proof of (7) by mathematical induction, not involving Theory of Equations, will be found in the *Edin. Math. Soc. Proceedings*, XIX., p. 36. R. F. MUIRHEAD.

MATHEMATICAL NOTES.

124. [D. 2. b.] Addition Series.

An Addition series is one in which $u_n = u_{n-1} + u_{n-2}$, the principal addition series being

$$1, 1, 2, 3, 5, 8, 13, 21, \text{etc.}, \dots \text{A.}$$

which was originally invented to obtain numbers by which to divide, practically, a line in extreme and mean ratio. In fact, in A.,

$$u_n = u_{n-1} + u_{n-2}, \text{ and } u_n u_{n-2} - u_{n-1}^2 = \pm 1,$$

which is as near an approximation to E. and M. ratio as is possible in integers.

Here,
$$u_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\};$$

and, by induction, the sum of n terms $= u_{n+2} - 1$.

The following are properties of A. which can be proved by induction, or from the value of u_n .

If $M = u_n u_{n-3} - u_{n-1} u_{n-2}$, then $M = \pm 1$ (M in G.P. and A.P. respectively being 0 and $-2b^3$).

$$u_0 = 0; \text{ and } u_{-n} = \pm u_n.$$

The mn^{th} term is divisible by the n^{th} .

$$u_n = u_m u_{n-m+1} + u_{m-1} u_{n-m}.$$

$u_n^2 - u_{n-2}^2$ and $u_n^2 + u_{n+1}^2$ are each equal to a term of A.

One of the two $u_n^2 \pm u_m^2$ is always equal to the product of two terms of A.

The general Addition series can be expressed in terms of A., for it is :

$$a, b, a+b, a+2b, 2a+3b, 3a+5b, \text{etc.} \dots \text{G.A.}$$

The sum of n terms of G.A. is $au_n + b(u_{n+1} - 1)$ where u_n is a term of A.

In G.A., $M = \pm(a^2 + ab - b^3)$; and $U_n U_{n-2} - U_{n-1}^2 = M$; where U_n is a term of G.A.

The sum of n terms is $U_{n+2} - U_2$; and the sum of n terms after the m^{th} is $U_{n+m+2} - U_{m+2}$.

Also

$$U_{2n+1} = (1 + 1) \prod_{i=1}^n U_i U_2 U_3 \dots U_{n+1}.$$

In the series $a, b, a+b, 2a+b, 3a+2b, 4a+3b$, etc.,
in which U_n, U_{n+1}, U_{n+2} is an A.P. if n is even, and a G.A. if n is odd,

$$M = \pm(2a^2 - b^2), \text{ and } U_n = 2U_{n-2} + U_{n-4}$$

In the series composed of two G.A. placed alternately

$$a, x, b, y, a+b, x+y, a+2b, x+2y, \text{ etc.},$$

M = alternately $\pm(ag - bx)$ and $\pm\{a+bx - by\}$; thus $M = \pm 1$, in

$$2, 1, 3, 2, 5, 3, 8, 5, 13, 8, \text{ etc.}$$

M is, in turn, zero and $\pm(\text{a term of A.})^2$ in the following series in which $u_1, u_3, u_5, \text{ etc.}$, is A.; and also $u_2, u_4, u_6, \text{ etc.}$; and $u_3, u_6, u_9, \text{ etc.}$; viz.:

$$1, 2, 3, 2, 3, 5, 3, 5, 8, 5, 8, 13, \text{ etc.}$$

It is seen from

$$1, r, r^2, r+r^2, (1+r)^2, (1+3r+2r^2), \text{ etc.},$$

that it is possible to have a series of integral functions in which u_n, u_{n+1}, u_{n+2} is a G.P. if n is odd, and a G.A. if n is even. W. H. LAVERY.

125. [K. 20. d.] *Approximation to tan A.*

Up to 10° the ordinary formula, $\tan A^\circ = \cdot 0175A$, gives close enough results for many purposes; but from 10° to 25° or so it is better to use

$$\tan A = \frac{A}{60} + \frac{A}{6000} + \left(\frac{A}{120}\right)^2.$$

Three examples are given.

$A = 10^\circ 8$	$16^\circ 7$	$24^\circ 6$
$60 \overline{) 10 \cdot 8}$	$60 \overline{) 16 \cdot 7}$	$60 \overline{) 24 \cdot 6}$
$\cdot 1800$	$\cdot 2783$	$\cdot 4100$
18	28	41
81	196	420
$\tan A = \cdot 1899$	$\cdot 3007$	$\cdot 4561$
From tables, $\cdot 1908$	$\cdot 3000$	$\cdot 4578$

are seen to be the more accurate values.

The error is usually less than that in the last example, and as far as I have tried, the calculation gives the true tangent of some angles within a very few minutes of the given one.

An approximate reverse formula is

$$A = 61 \tan A - 16 \tan^2 A,$$

which may be applied to any angle up to 45° with fair exactness, the maximum error in the value of A so found being about a quarter degree.

I have been told that the latter has already appeared in some book on Yachting. C. E. M'VICKER.

126. [V. a.; K. 20. a.] *Note on the Teaching of "Solution of Triangles" in Trigonometry.*

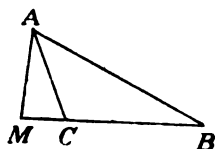
The methods commonly described in text-books for the logarithmic solution of oblique-angled triangles do not permit of the subject being adequately studied until a knowledge has been acquired of the properties of multiple and submultiple angles, and the so-called semi-sum and semi-difference formulae of Trigonometry. The object of the present note is to show how a beginner may proceed directly to the logarithmic solution of oblique-angled triangles with no knowledge beyond what is involved in the definitions of the trigonometric functions, the use of logarithmic tables, together with a preliminary drilling in solution of right-angled triangles.

The cases where a side and two angles or two sides and an angle opposite one are given present no difficulty, as they are solved by the sine rule which can be easily proved. The remaining cases are those in which two sides and the included angle or three sides are given.

For the first of these cases let CA, CB be the two given sides, BCA being the included angle, and suppose CB is the greater of the two sides (so that we do not have to consider the possibility of B being obtuse).

Drop AM perpendicular on BC .

Then we have



if C be acute $MA = b \sin C,$ $CM = b \cos C,$ $BM = a - CM.$	if C be obtuse $MA = b \sin (180^\circ - C),$ $CM = b \cos (180^\circ - C),$ $BM = a + CM.$
---	--

$$\tan B = \frac{AM}{BM'}$$

and

$$c = \frac{AM}{\sin B} \quad \text{or} \quad c = \frac{BM}{\cos B}$$

Finally,

$$A = 180^\circ - B - C.$$

For the second case let a be the greatest of the three sides (so that we do not have to consider the possibility of either B or C being obtuse), and suppose $b > c$.

Drop AM perpendicular on BC as before.

Then, by I. 47,

$$CM^2 - BM^2 = CA^2 - BA^2 = b^2 - c^2,$$

and

$$CM + BM = a.$$

Hence

$$\log (CM - BM) = \log (b + c) + \log (b - c) - \log a.$$

This finds CM and BM , and then the angles are given by the formulae

$$\cos B = \frac{BM}{c}, \quad \cos C = \frac{CM}{b}.$$

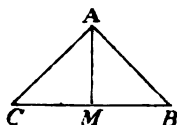
The following examples will show how simply and easily triangles can be solved by these methods with a book of four-figure tables.

Ex. 1. $a = 684$, $b = 504$, $C = 94^\circ 16'$.

Here C is obtuse, and $CM = b \cos 85^\circ 44'$.

		Log.	Antilog.	(1)
	b	2.7024		(1)
	$\cos (180^\circ - C)$	8.8717		(2)
	$\sin (180^\circ - C)$	9.9988		(3)
From (1), (2),	CM	1.5741	37.51	(4)
From (1), (3),	MA	2.7012	502.4	(5)
	BM	2.8582	721.51	(6)
From (5), (6),	$\tan B$	9.8430	$B = 34^\circ 52'$	(7)
	$\sin B$	9.7577		(8)
From (5), (8),	AB	2.9441	$c = 879.2$	(9)
Finally,	$A = 180^\circ - 94^\circ 16' - 34^\circ 52' = 50^\circ 52'.$			

Ex. 2. $a=45.2$, $b=32.9$, $c=15.4$.



	No.	Log.
b	32.9	
c	15.4	
$b+c$	48.3	1.6839
$b-c$	17.5	1.2430
$b^2 - c^2$		2.9269
$CM + BM = a$	45.2	1.6551
$CM - BM$	18.70	1.2718
$2CM$	63.90	
$2BM$	26.50	

	No.	Log.		No.	Log.
CM	31.95	1.5045	BM	13.25	1.1223
$CA (=b)$	32.9	1.5172	$BA (=c)$	15.4	1.1875
$\cos C$		9.9873	$\cos B$		9.9347

$\therefore C = 13^\circ 47'$

$B = 30^\circ 38'$

Finally, $A = 180^\circ - 13^\circ 47' - 30^\circ 38' = 135^\circ 35'$.

The solutions of these two examples by conventional methods would be as follows:

Ex. 1.			Ex. 2.		
	No.	Log.		No.	Log.
a	684		a	45.2	
b	504		b	32.9	
$a-b$	180	2.2553	c	15.4	
$a+b$	1188	3.0748		2) 93.5	
		1.1805	s	46.75	1.6693
$\cot \frac{1}{2}C$	$\cot 47^\circ 8'$	9.9676	$s-a$	1.55	0.1903
$\tan \frac{1}{2}(A-B)$		9.1481	$s-b$	13.85	1.1415
$\frac{1}{2}(A-B) =$	$8^\circ 0'$		$s-c$	31.35	1.4962
$\frac{1}{2}(A+B) =$	$42^\circ 52'$		$(s-a) \div s$		2.5305
$A =$	$50^\circ 52'$		$(s-c) \div (s-b)$		.3567
$B =$	$34^\circ 52'$		$\tan^2 \frac{1}{2}B$		2.8862
$a+b$		3.0748	$\tan^2 \frac{1}{2}C$		2.1748
$\sin \frac{1}{2}C$		9.8651	$\tan \frac{1}{2}B$	$\tan 15^\circ 30'$	9.4437
$\cos \frac{1}{2}(A-B)$		12.9399	$\tan \frac{1}{2}C$	$\tan 6^\circ 58'$	9.0874
		9.9958	B	$31^\circ 0'$	
c	879.2	2.9441	C	$13^\circ 56'$	

Finally, $A = 180^\circ - 31^\circ 0' - 13^\circ 56' = 135^\circ 4'$.

A comparison of the alternative methods shows that there is no appreciable saving of labour in the use of the elaborate formulae commonly taught, and that by the use of the simpler methods here suggested the subject may be taught to beginners at an earlier stage than has been hitherto customary.

G. H. BRYAN.

REVIEWS.

A Course of Modern Analysis. By E. T. WHITTAKER, M.A. Pp. xvi. 378. (Cambridge University Press. 1902.)

This work is sure of a favourable reception because it gives in a moderate compass an attractive account of some of the most valuable and interesting results of recent analysis. The first part deals mainly with infinite series, especially

power-series and Fourier expansions: it also includes the elements of complex integration and of the theory of residues. The second part comprises chapters on the gamma-function, Legendre functions, the hypergeometric series, Bessel functions, the equations of mathematical physics, and elliptic functions.

The author has secured brevity by adopting, on the whole, a deductive method of exposition. The advantages of this are shown very clearly in the chapter on the gamma-function. By taking Weierstrass's definition of $\Gamma(z)$ as an infinite product, the properties of the function are deduced with extreme simplicity, and although the historical order of development is almost reversed, we feel that the new treatment is really the true and natural one. The gamma-function is, in fact, the simplest case of a transcendental function with an infinite number of simple poles, and this property is made obvious by the new definition. (It is curious, by the bye, to see how very nearly Weierstrass was anticipated in this matter by Gauss.) Again, the new method has suggested interesting extensions to Alexeiewsky, Barnes, and others, so that the change of view has amply justified itself.

On the other hand, Mr. Whittaker's chapter on the hypergeometric series seems to me to illustrate the risk of becoming artificial, which is incurred by an author who sacrifices induction overmuch. Here, for instance, we have Papperitz's form of the differential equation satisfied by the most general form of Riemann's P -function set down without any explanation of the method by which it is obtained. For all the reader can tell, it might have dropped from the skies, or have been written on a wall by some spirit from another world. It has not entered into the plan of the treatise to give any general account of linear differential equations: but in this context it would have been possible to bring out the essential fact that the equation defines a function with three critical points, each associated with two indices. The suppression of this fundamental property obscures the real meaning of much that follows, and gives an air of hocus-pocus to the demonstrations. The principal vice of Cambridge methods and Cambridge text-books has always been the encouragement of students to take their mathematics ready-made, to assimilate facts without inquiring into their sources. By some of us, at least, it was hoped that the institution of Part II. of the Mathematical Tripos would help to counteract this tendency. Alas! the conscientious ingenuity of tutors and lecturers, aiming at immediate results, is likely to defeat the main purpose of the innovation, and produce a state of things worse than that under the old régime.

But to return to Mr. Whittaker's book. One very good feature is that the relations of the functions of Legendre, Bessel, etc., to the general hypergeometric function have been brought out. If Chaps. x., xi. had been transposed this might perhaps have been done even more effectively. Such generalisations as this are the direct result of recent function-theory, and help to lighten the burden with which the progress of mathematics appears to threaten us. Apart from special subtleties, which appeal to the adept, the general tendency of function-theory is towards simplicity and clearness. There is no reason, for instance, why a school-boy should not learn the proper expression for $\sin x$ as an infinite product, or the elementary theory of power-series for a complex variable. New ideas are not difficult simply because they are new: if there is something which I never heard of until I was fifty, it does not follow that no one else is to hear of it before reaching the same age. And why persist in putting before boys and girls ideas which are positively erroneous, and methods entirely out of date?

This same gain of simplicity is illustrated by Mr. Whittaker's chapters on elliptic functions. It is not sufficiently shown why the ratio of the periods should be non-real; and, personally, I should prefer the omission of proofs of the addition-theorem which are not straightforward applications of Abel's method. But with these exceptions the discussion, as far as it goes, is very clear; and apart from the theories of transformation and complex multiplication (the omission of which is natural enough), the reader will find all the facts he is likely to require. One reflection suggested by reading this and other parts of the book is that the reputation of Cauchy and Liouville are likely to be enhanced as time goes on. It seems to me that even yet their contributions to science are insufficiently recognised. In this connection it may be remarked that Mr. Whittaker gives, besides Dirichlet's proof of the Fourier expansions, Cauchy's second demonstration (published in 1827) with some necessary amendments.

In some respects, I think, this treatise improves as it goes on. The first two or three chapters are neither so agreeably nor so accurately written as those which follow, and there is an occasional want of proportion. For instance (p. 12) "Series whose convergence is *due* to the convergence of the series formed by the moduli of their terms . . . are called absolutely convergent series" (the italics are mine): again, it is assumed (p. 13) that u^{n+1} tends to zero for $n=\infty$ when $|u| < 1$, although facts quite as obvious, or more so, are proved in detail; on p. 19 the statement

"As n increases, v_{n+1}/v_n will therefore tend to the limit

$$1 - (1 + \frac{1}{2}c)/n$$
"

is not satisfactory; while the second part of Art. 6 seems to me to obscure a very simple matter. Finally, I confess that I cannot follow all the argument of Arts. 84, 85: this is probably my own fault.

Of casual slips and misprints there are apparently few. It is, of course, wrong to say that $(z-1)!$ is a polynomial (p. 209); on p. 24 $(u+p)$ should be $(n+p)$: p. 46 for "excluded" read "included"; p. 201 for "Alexerewsky" read "Alexiewsky."

G. B. MATHEWS.

Practical Exercises in Geometry. By W. D. EGGAR, M.B. Macmillan & Co.

A New Geometry for Beginners. By R. ROBERTS, B.Sc. Blackie & Son.

Geometry. By S. O. ANDREW, M.A. John Murray.

Elements of Geometry. By R. LACHLAN, Sc.D. and W. C. FLETCHER, M.A. Edward Arnold.

Elementary Geometry. By W. M. BAKER, M.A. and A. A. BOURNE, M.A. G. Bell & Sons.

Plane Geometry. By T. PETCH, B.A. Edward Arnold.

Theoretical Geometry for Beginners. By C. H. ALLOOCK. Macmillan & Co.

Works on Geometry, whose appearance is due to alterations in the regulations of various examining bodies, continue to issue from the press. They may be divided roughly into two classes—(1) those which devote themselves to a course of Experimental Geometry, intended to precede and prepare for the future course of Deductive Geometry; (2) those intended to supply that future course, and replace the editions of Euclid's Elements, modernized or otherwise, now current in the schools.

There seems more immediate need of the former, the experimental courses throwing a great strain on teachers who have to deal with many other subjects besides Geometry, and have neither the time nor the inclination to develop a systematic course of experimental work. Excellent as are the little works by Mault, Bert, and Spencer, something more is wanted, and we welcome heartily Mr. Eggar's book as one that should supply a widely-felt want. It can be used both by those who are preparing their pupils for, and by those who are taking their pupils through, a deductive course. We hope it will be used largely by teachers of both categories, for we hold that experimental methods should accompany, as well as precede, deduction. How much vividness a theorem in loci gains, for instance, by such experimental treatment as that given on pp. 170, 186! The book is well got up; the figures are effectively drawn on an ample scale. We see everywhere signs of a teacher whose heart is in his work, and whose efforts, we feel assured, will help to kindle enthusiasm, not only among those who are under his direct personal influence, but among those also who are trained after the model he has set.

Mr. Roberts' work is of an intermediate character. Though without special experience of their needs, we imagine it excellently suited for technical schools. It is a rapid course through the essentials of Geometry. Theorems and problems seem judiciously chosen, and calculated to interest the student and lead to further studies and applications. We can best explain how condensed the scheme of treatment is by the statement that out of the 87 pages of which it consists, 5 are devoted to graphs and curve tracing, and that in the other 82 the author treats the subject matter of the ordinary school Euclid both theoretically and practically. The diagrams are numerous and well drawn; the five pages on curve tracing are

illustrated by 8 full-page diagrams, whose black lines stand out effectively on the tawny yellow of the millimetre ruled paper. We fancy it will be found useful by many teachers, besides those who use it as a class book. We could find fault with some of the definitions, but we could find the same, or greater, with those of a preceding writer, whose merits have been held to cover them. Mr. Andrew's work is of a similar character, his course being more extensive. In 178 pages he not only gets through a course of Plane and Solid Geometry, but also treats of Graphs, Mensuration, Trigonometry, and Descriptive Geometry. It seems likely to be very useful to teachers. The course is well chosen, the diagrams and type excellent. The remaining four books belong to the second class, and differ from each other chiefly in the extent of the course given, and in fulness of treatment. All are good examples of typographical excellence, which is more common now than in the days of Potts and Todhunter. Any one of them in the hands of a good teacher would be found an excellent basis for a course of lessons. Messrs. Lachlan & Fletcher in 204 pages give a course of the Elements of Plane Geometry, with two useful sections dealing with Geometrical Analysis and Maxima and Minima. They evade the difficulties of parallels by use of the "direction." Ratio and Proportion, and the Circle precede Areas. There is a numerous set of exercises, and illustrations from Mensuration and Trigonometry are appropriately given. Messrs. Baker and Bourne in 272 pages get through the essentials of Euclid's four books. The treatment on the whole is conservative. "Playfair's Axiom" replaces Euclid's celebrated postulate on parallels. Algebraical proofs are given of the principal relations between the segments of a line, and Arithmetical examples given throughout. Nine pages are devoted to Graphs.

Mr. Petch in 112 pages gets through a serviceable course, extending to Similarity. Mr. Allcock devotes 135 pages chiefly to the subject matter of Euclid's First Book. There are sections on the numerical treatment of Area, Analyses and Loci Parallel are treated by Playfair's Axiom. A convenient list of Definitions, Axioms, and Postulates is given at the end.

The Essentials of Plane and Solid Geometry, by WEBSTER WELLS, S.B. (D. C. Heath & Co., Boston.)

Mr. Wells has arranged his *Essentials* on much the same plan as the shorter English treatises noticed above, but extends to spherical Geometry. It seems an excellent treatise. Its scope is much the same as that of the revised edition of Beman and Smith's *Plane and Solid Geometry*. We do not know if there is any English work published in England of the same range. Doubtless such will not be long in making their appearance. English authors may well take the *designs* and English firms the *execution* of diagrams in these books for models. There is a dainty grace about them which ours seem to lack.

High School and Academic Algebra, by LOUIS PARKER JOCKEY, (Butler, Sheldon & Co., New York.)

This seems a clearly written and carefully arranged treatise. It extends to Permutations and Combinations and devotes sections to Logarithms (Practical), Undetermined Coefficients, Partial Fractions, and Determinants. There are numerous model solutions and large collections of exercises. The author lays great stress on 'checking' of solutions.

Advanced Algebra and Geometry, by WILLIAM BRIGGS, LL.D., M.A. (W. B. Clive.)

This is best characterised by its sub-title, as the Algebra and Geometry required for the Syllabus in Advanced Mathematics of the London University Matriculation Examination. It can be heartily recommended for its purpose.

EDWARD M. LANGLEY.

A Graphic Method for solving certain Questions in Arithmetic or Algebra, by G. L. VOSE. 2nd Edition. Pp. 62. 1902. (Van Nostrand, New York.)

The methods described in this little brochure were originally devised for railway purposes. How difficult such problems can be, may be guessed from a glance at train connections on a time-table. Perhaps some of our readers may like to amuse themselves by constructing a graphical solution of the following

simple case. A man walks round an island at the rate of five miles an hour, and at the same time another starts at such a rate as to take him round its ten mile circumference in three and one-third hours. When will the first pass the second, and when will he pass him the second time? We have found that boys are fond of these games.

Annuaire pour l'An 1903. Published by the Bureau of Longitudes. Pp. 668, 140. 1903. 1 fr. 50 c. (Gauthier-Villars.)

This little volume maintains its reputation for containing more matter than probably any other book in the world. So vast is the amount of astronomical, physical, geographical, and statistical information it contains, that, as the preface states, "il est clair qu'on ne saurait aller plus loin." Accordingly a re-arrangement of the contents has become imperative. Tables that never or seldom vary will not be yearly repeated, but the reader may count on getting from any two consecutive yearly numbers the tables he requires. Apart from the collection of tables, the present volume contains essays on shooting stars and comets by M. Radau; on Science and Poetry by the veteran astronomer, M. Janssen; the funeral orations on MM. Cornu and Faye; and an account of M. Janssen's work in 1902 at the observatory on Mont Blanc. The editors of the *Annuaire* are MM. Janssen, Lippman, Loewy, and last but not least, M. Poincaré. We cannot refrain from quoting an anecdote told by M. Janssen, expressing the awe which is felt by minds of a higher order on being first confronted with a striking instance of the laws which govern the universe. When Janssen was explaining to his pupil Gounod the laws of Kepler, "Gounod s'écria: 'Ah! que c'est beau!' et des larmes lui montèrent aux yeux."

Récréations Arithmétiques, by E. FOURREY. 2nd Edition. 1901. (Nony.)

This interesting collection of amusing arithmetical conundrums will make a capital birthday book for a boy in whom we wish to cultivate a taste for enquiry. They require but an elementary knowledge of the rules of arithmetic. The first part deals with the operations of the art, progressions, polygonal numbers, squares, cubes, divisors, and the simpler properties of numbers. The applications comprise such exercises as finding the day of the week corresponding to a given date, card tricks, the "think of a number" type, and oh! l'éternel féminin! the "three jealous husbands crossing a river with their wives, the boat available holding but two at a time"; and the like. The third part consists of an admirable introduction to the mysteries of squares, magic, diabolical and hypermagical, magic cubes and magic circles.

I. Tre Problemi Classici degli Antichi:—Prob. I. **La Quadrature del Cerchio**, by Prof. BELLINO CARRARA S.J. Pp. 168. Reprinted from the *Revista di Fisica, Matematica e Scienze Naturali*. 1902. (Fusi, Pavia.)

In this pamphlet Prof. Carrara takes as the first of the three great problems of antiquity that of the squaring of the circle, and gives the reader a full historical and critical analysis of the attempts that have been made to solve it. To those who can read Italian, this will be found a most convenient summary. The style is easy, and references are copious. The various stages in the attempt to determine the ratio of the diameter to the circumference are clearly sketched, and the influence of the growth of modern analysis is skilfully and lucidly traced. To many the most interesting portion of Prof. Carrara's labours will be the pages in which he sets forth the progress of investigation from the advent of Hermite's *Sur la Fonction exponentielle*, 1873, and the gradual and successive simplification of the researches of that writer and Lindemann by Weierstrass, Hilbert, Hurwitz, and Gordan. Gordan's treatment is, in fact, quite elementary. It may be found in an English dress in chaps. iii. and iv. of Part II. of Klein's *Famous Problems*, (Ginn and Co., 1897.) We look forward with interest to the appearance of the companion volumes which will deal with the Delian problem and that of the trisection of an arbitrary angle.

The Schoolmasters' Year Book and Directory. 1903. (Swan Sonnenschein.)

We break our general rule in drawing the attention of readers of the *Gazette* to the appearance of this extremely handy volume. It contains lists with necessary

details of all the educational societies and organisations in the country, together with a vast amount of information useful to teachers which cannot be found within the covers of any other publication with which we are acquainted. Part III. contains a series of articles and reviews, including a notable article by Prof. Minchin on "The Reform of Mathematical Teaching." We cordially greet the advent of the *Year Book*. As for the *Directory*, we cannot agree with the Head Master who thought "the whole idea absolutely rotten." On the contrary, it will play its part in the raising the status of the profession.

A Treatise on Differential Equations. By A. R. FORSYTH. Third Edition. Pp. 511. 14s. 1903. (Macmillan.)

Good wine needs no bush. Professor Forsyth's treatise on Differential Equations remains a classic. It is the most lucid, accurate, and exhaustive exposition of the subject in our language. It is a matter of pride to English mathematicians to know that it had been translated into both German and Italian. Students may be interested to know that Dr. Maser's translation, published at Brunswick in 1889, contains 256 pages of solutions to the questions. To be up to date, Dr. Maser will have to furnish solutions to between two to three hundred more questions, which are appended to the new edition. More than eighty pages of additional matter are included in the volume before us. A sketch is given of Runge's method for the numerical solution of ordinary differentials, limited to insoluble equations of the first order (*Math. Ann.* xlv., 1895, pp. 168-178). We also find an outline of the method for the integration of linear equations in series which is connected with the name of Frobenius (*Crelle*, lxxvi., 1873, pp. 214-224, and Forsyth's *Theory of Diff. Eq.*, vol. iv., pp. 78 et seq.). This, again, applying to equations of any order, is confined to those of the second order, with special applications to Bessel's equations. A welcome addition is the introduction to Jacobi's *Theory of Multipliers*, pp. 319-328.

PROBLEMS.

481. [L¹. 10. d.] PQ is a chord of a parabola meeting the axis in a fixed point H so that $AH = 8AS$, and $P'Q'$ its projection on the tangent at the vertex: shew that the locus of the intersection of PQ' . $P'Q$ is a circle.

E. N. BARISIEN.

482. [J. 1. d.] Each sheet of a book is folded n times so as to make 2^{n+1} pages. Investigate the way in which the different pages must be arranged in the forms when the type is put into pages.

G. H. BRYAN.

483. [L¹. 4. c.] The tangents to an ellipse at Q and R intersect at a point P on a coaxial ellipse whose axes are respectively n times those of the former curve. Find the locus of the centroid of the triangle PQR .

A. F. VAN DER HEYDEN.

484. [K. 22. d.] The pole of a small circle on the earth is in latitude l and longitude λ and the angular radius of the circle is α . Find the equation of its stereographic projection (the eye being in the equator in longitude 180°).

C. S. JACKSON.

485. [M¹. 3. 1. β .] P' is the centre of curvature at P , a point on

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = (2a)^{\frac{1}{2}}.$$

Find the locus of the intersection of OP' with the tangent at P , O being the origin.

V. RETALI.

486. [I. 3.] Are there any values of p for which $2^{p-1} - 1$ is divisible by p^2 ?

D. O. S.

487. [R. 7. f.] A particle is tied to a fixed point by a string whose natural length is a , and whose extension is proportional to the tension; when the particle is in equilibrium with the string vertical the extension is c : prove

that when the string revolves as a conical pendulum about the vertical, making n revolutions per second, the extension is

$$\frac{4\pi^2 n^2 a c}{(g - 4\pi^2 n^2 c)} \quad (C.)$$

488. [R. 4. a. β .] A rough uniform rod (coefficient of friction $\frac{1}{2}$) rests over the edge of a cylindrical bucket of diameter 9 inches, with its lower end against the vertical side of the bucket, and when it is on the point of sliding down it makes with the vertical an angle $\tan^{-1} \frac{3}{4}$: shew that its length is 40 inches. (C.)

489. [R. b. γ .] Particles are projected with given velocity from a point at a distance a from a vertical wall so as just to clear the horizontal top of the wall: prove that their points of impact on a vertical plane parallel to the wall and at a distance $2a$ from the point of projection lie on a circle. (C.)

490. [R. 4. c.] A framework formed of five light bars AB, BC, CD, DA, AC , ΔBCD being a rectangle, is in equilibrium under the action of forces at the angles whose components in directions AB, BC respectively are X_1, X_2, X_3, X_4 , and Y_1, Y_2, Y_3, Y_4 . Shew that the stress in the diagonal bar AC is

$$(X_3 + X_4) \frac{AB}{AC} + (Y_2 + Y_3) \frac{BC}{AC}. \quad \text{Melbourne, 1901.}$$

SOLUTIONS.

338. [D. 2. b. β .] If $\theta + \phi + \psi = 4\pi$,

$$\sum_0^{\infty} \left[(-1)^r \frac{\sum \theta^{2r}}{2^r} \right] + 1 \\ = 4 \prod_0^{\infty} \left[1 - \frac{\sum \theta^2}{\pi^2} \cdot \frac{1}{(2r+1)^2} + \frac{\sum \theta^2 \phi^2}{\pi^4} \cdot \frac{1}{(2r+1)^4} - \frac{\theta^2 \phi^2 \psi^2}{\pi^6} \cdot \frac{1}{(2r+1)^6} \right].$$

L. WALLIS.

Solution by ANON.

$$\begin{aligned} & \sum_0^{\infty} \left\{ (-1)^r \frac{\theta^{2r} + \phi^{2r} + \psi^{2r}}{2^r} \right\} + 1 \\ &= 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4} - \frac{\theta^6}{6} + \dots \\ & \quad + 1 - \frac{\phi^2}{2} + \dots \\ & \quad + 1 - \frac{\psi^2}{2} + \dots + 1 \\ &= \sum \cos \psi + 1 \\ &= 2 \cos \frac{\theta + \phi}{2} \cos \frac{\theta - \phi}{2} + 2 \cos^2 \frac{\psi}{2} \\ &= 2 \cos \frac{\psi}{2} \left\{ \cos \frac{\theta + \phi}{2} + \cos \frac{\theta - \phi}{2} \right\} \\ &= 4 \prod \cos \frac{\psi}{2} \\ &= 4 \prod_0^{\infty} \left(1 - \frac{\theta^2}{\pi^2 (2r+1)^2} \right) \prod_0^{\infty} \left(1 - \frac{\phi^2}{\pi^2 (2r+1)^2} \right) \prod_0^{\infty} \left(1 - \frac{\psi^2}{\pi^2 (2r+1)^2} \right) \\ &= 4 \prod_0^{\infty} \left\{ 1 - \frac{\sum \theta^2}{(2r+1)^2 \pi^2} + \frac{\sum \theta^2 \phi^2}{(2r+1)^4 \pi^4} - \frac{\theta^2 \phi^2 \psi^2}{(2r+1)^6 \pi^6} \right\}. \end{aligned}$$

382. [D. 2 d.] Prove that $(\Sigma p_n)^2 + (\Sigma p_n + 1)^2 = (p_{2n} + q_{2n})^2 = q_{2n+1}^2$, where Σp_n signifies the sum of the even convergents of $p^2 - 2q^2 = \pm 1$.

$$[\text{Ex. } n=3. \quad 119^2 + 120^2 = (99 + 70)^2 = 169^2.]$$

R. W. D. CHRISTIE.

Solution.

$$\sqrt{2} = 1 + \frac{1}{2} + \frac{1}{2} + \dots;$$

$$\therefore 1 + \frac{p_{n+1}}{q_{n+1}} = 2 + \frac{1}{2} + \frac{1}{2} + \dots \quad (n+1 \text{ terms})$$

$$= 2 + \frac{1}{1 + \frac{p_n}{q_n}}$$

$$= \frac{2p_n + 3q_n}{p_n + q_n}.$$

Hence

$$q_{n+1} = p_n + q_n.$$

Again,

$$\begin{aligned} p_{2n+1} &= 2p_{2n} + p_{2n-1} \\ &= 2p_{2n} + 2p_{2n-2} + p_{2n-3} \\ &\dots\dots\dots \\ &= 2\Sigma p_n + p_1; \end{aligned}$$

$$\therefore \Sigma p_n = \frac{1}{2}(p_{2n+1} - 1),$$

and

$$\begin{aligned} (\Sigma p_n)^2 + (\Sigma p_n + 1)^2 &= \frac{1}{2}(p_{2n+1}^2 + 1) \\ &= q_{2n+1}^2 = (p_{2n} + q_{2n})^2. \end{aligned}$$

410. [L. 4. c; M. 3. b.] On the tangent at M to an ellipse is measured off a length MN equal to the distance of the tangent from the centre of the ellipse: prove that the locus of N is a unicursal whose area is $\frac{\pi}{2}(a+b)^2$.

E. N. BARISIEN.

Solution by PROPOSER and H. G. BELL.

If ϕ be eccentric angle of M ,

$$MN = p = \frac{ab}{\sqrt{b^2 \cos^2 \phi + a^2 \sin^2 \phi}},$$

and makes with transverse axis an angle $\omega = \tan^{-1} \left(\frac{b}{a} \cot \phi \right)$.

Hence the coordinates of N are

$$a \cos \phi + \frac{a^2 b \sin \phi}{a^2 \sin^2 \phi + b^2 \cos^2 \phi}, \quad b \sin \phi - \frac{ab^2 \cos \phi}{a^2 \sin^2 \phi + b^2 \cos^2 \phi},$$

and the locus of N is unicursal.

The area outside ellipse

$$\begin{aligned} &= \int_0^{2\pi} \frac{1}{2} p^2 d\omega = \frac{1}{2} \int_0^{2\pi} (a^2 \cos^2 \omega + b^2 \sin^2 \omega) d\omega \\ &= \frac{\pi}{2} (a^2 + b^2), \end{aligned}$$

and the whole area is $\frac{\pi}{2} (a^2 + b^2) + \pi ab = \frac{\pi}{2} (a+b)^2$.

412. [L¹. 2; 14. a.] *E* is a conic circumscribing the triangle *ABC*, and its centre *O* lies in a fixed straight line: shew that the poles of the sides of *ABC* with respect to *E* lie on three conics which cut the corresponding sides in points on a fourth conic.

E. P. EVANS.

Solution by W. F. BEARD.

Let the centre locus meet the sides of *ABC* in *L*, *M*, *N*, and take on these sides *L*₁, *M*₁, *N*₁, harmonic conjugates of *L*, *M*, *N*.

Then *LM*₁*N*₁, *L*₁*MN*₁, *L*₁*M*₁*N* are straight lines, and the polars of *L*, *M*, *N* with respect to *E* are parallel lines through *L*₁, *M*₁, *N*₁: let them meet *LMN* in *L*₂, *M*₂, *N*₂, and let *A'* be the mid-point, and *P* the pole of *BC*, so that *P* is the intersection of *OA'* and *L*₁*L*₂. Then

$$P\{NA'ML_1\} = \{NOML_2\} = \{N_2 \propto M_2L\}$$

(since *LL*₂, *MM*₂, *NN*₂ form involution, centre *O*)

$$= \frac{N_2L}{N_2L} = \frac{M_1L}{N_1L},$$

which is constant. Therefore the locus of *P* is a conic through *A'*, *L*₁, *M*, *N*, and since *AA'*, *BB'*, *CC'*, and *AL*₁, *BM*₁, *CN*₁ are concurrent triads, *A'*, *B'*, *C'*, *L*₁, *M*₁, *N*₁ all lie on one conic.

Solution by R. F. DAVIS and C. V. DURELL.

Let $\Sigma l\beta\gamma = 0$ be the trilinear equation of the conic circumscribing *ABC*. The coordinates of the centre are proportional to $l(-al + bm + cn)$, ... , and the centre always lying on a fixed straight line $\Sigma P^2 - 2\Sigma Pmn = 0$. The coordinates of the pole of *BC* are $(-l, m, n)$, and it therefore lies on the conic $\Sigma Pa^2 - 2P'\gamma a + 2R'a\beta = 0$, which conic intersects *BC* in the same points as the conic $\Sigma Pa^2 - 2\Sigma P'\beta\gamma = 0$.

Mr. Davis solves also by projecting *B*, *C* into the focoids.

417. [L¹. 5; 17. e.] *Eight normals to an ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *touch any concentric co-axial ellipse* $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$. *Prove this, and find the conditions for reality.*

Trip., 1895.

Solution by W. F. BEARD, R. F. DAVIS and C. V. DURELL.

$\lambda x + \mu y = 1$ is a normal to first ellipse if

$$a^2/\lambda^2 + b^2/\mu^2 = (a^2 - b^2)^2,$$

and a tangent to the second if

$$a'^2\lambda^2 + b'^2\mu^2 = 1.$$

Hence, eliminating λ , we have

$$b'^2(a^2 - b^2)^2\mu^4 + \mu^2\{a^2a'^2 - b^2b'^2 - (a^2 - b^2)^2\} + b^2 = 0, \dots\dots\dots A$$

giving four values of μ , for each of which there are two values of λ , giving in all eight solutions.

If all the solutions are real, both values of μ^2 given by A must be real and positive. Hence

$$(a^2a'^2 - b^2b'^2)^2 - 2(a^2 - b^2)^2(a^2a'^2 + b^2b'^2) + (a^2 - b^2)^4 > 0,$$

$$\text{i.e.} \quad \{(a^2 - b^2)^2 - (aa' + bb')^2\} \{(a^2 - b^2)^2 - (aa' - bb')^2\} > 0;$$

and also

$$(a^2 - b^2)^2 > (a^2a'^2 - b^2b'^2);$$

and both these conditions are satisfied if

$$a^2 - b^2 > aa' + bb',$$

and this ensures also that the values of λ^2 are real and positive.

424. [R. 1. a.] *Two points moving at equal speeds, one on a circle, the other on a tangent, arrive at the point of contact simultaneously. Where does the line joining them meet the diameter through the point of contact just before they coincide?*
C. E. M'VICKER.

Solution by J. BLAIKIE, J. F. HUDSON, A. W. POOLE, and PROPOSER.

Let O be the centre of the $\odot$, P , T two positions of the moving points and let TP meet the diameter through A , the point of contact in X , and let $POA = \theta$. Then

$$\frac{AX}{AT} = \frac{AX - NX}{AT - PN} = \frac{OA - ON}{AP - PN} = \frac{1 - \cos \theta}{\theta - \sin \theta} = \frac{\frac{\theta^2}{3} - \frac{\theta^4}{5} + \dots}{\frac{\theta^3}{3} - \frac{\theta^5}{5} + \dots};$$

$$\therefore AX = AO \frac{3 - \frac{\theta^2}{4} + \dots}{1 - \frac{\theta^2}{20} + \dots} = 3AO, \text{ when } \theta = 0.$$

The result may be deduced from Snell's approximation, $\theta = \frac{3 \sin \theta}{2 + \cos \theta}$.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor*. 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.).]

The Mathematical Monthly. Vols. I-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

(2) Wanted.

Vols. I-IV. *Mathesis*. 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

At the suggestion of Professor G. H. Bryan, F.R.S., this column is opened for the benefit of members of the Association. To members no charge will be made for insertion of requirements. Replies and enquiries should be addressed to the Editor when no postal address is attached to the items in the column.

BOOKS, ETC., RECEIVED.

Practical Exercises in Geometry. By W. D. EGGAR. Pp. xii., 287. 2s. 6d. 1903. (Macmillan.)

Elementary Arithmetic of the Octimal Notation. By G. H. COOPER. Pp. 70. n. p. 1902. (Whitaker Ray, San Francisco.)

Elements of Geometry. By R. LACHLAN and W. C. FLETCHER. Pp. xii., 208. 2s. 6d. 1903. (Arnold.)

Plane Geometry. By T. PETCH. Pp. v., 112. 2s. 1903. (Arnold.)

Intermediate Mathematics. University Tutorial Series. Sixth edition. 1902. (Clive.)

Récréations Arithmétiques. By E. FOURREY. 2nd. edition. Pp. 261. 8 fr. 1901. (Nony.)

- Key to "The Elements of Hydrostatics."* By S. L. LONEY, M.A. Pp. 146. 1902. (Cam. Univ. Press.)
- A Short Introduction to Graphical Algebra.* By H. S. HALL. 2nd edition. Pp. 49. 1903. 1s. (Macmillan.)
- Euclid V., VI., XI.* By RUPERT DEAKINS. Pp. 144. 1s. 6d. 1903. (Clive.)
- The Elementary Geometry of Conics.* By DR. C. TAYLOR, Master of St. John's College, Cambridge. Eighth edition revised, with a chapter on Inventio Orbium. Pp. viii., 156. 5s. 1903. (Deighton Bell.)
- Über die Maxwell-Hertz'sche Theorie.* By C. NEUMANN. No. II. Pp. 213-348. No. VIII. Pp. 755-860. From Vol. 27 der Abhandlungen der math.-physiken Classe der Königl. Sächsischen Gesellschaft der Wiss. 3 m. 50 pf. 1901, 1902. (Teubner.)
- De L'Expérience en Géométrie.* By C. DE FREYCINET. Pp. xix., 178. 4 fr. 1903. (Gauthier-Villars.)
- Leçons Élémentaires sur la Théorie des Fonctions Analytiques.* By E. A. FOUËT. Part I. Chaps. I.-V. Pp. 320. 7 fr. 50 c. 1902. (Gauthier-Villars.)
- Traité de Mécanique Rationnelle.* By PAUL APPELL. Vol. III.
- Équilibre et Mouvement des Milieux Continus.* By PAUL APPELL. Vol. III. of *Traité de Mécanique Rationnelle.* Pp. 560. 17 fr. 1903. (Gauthier-Villars.)
- Geometrie der Dynamen.* By E. STUDY. Part II. Pp. xiv., 241-603. 13 m. 40 pf. 1903. (Teubner.)
- Wahrscheinlichkeitsrechnung.* Part II. By E. CZUBER. Pp. 305-594. 12 m. 1903. (Teubner.)
- Problèmes de Mécanique.* By F. J. Pp. 574. 1903. (Ch. Poussielgue.)
- An Elementary Treatise on the Mechanics of Machinery.* By J. N. LE CONTE. Pp. x., 311. \$2.25. 1902. (The Macmillan Company.)
- The Thermodynamics of Heat Engines.* By S. A. REEVE. Pp. xi., 316, 42. 1903. (The Macmillan Company.)
- A Treatise on Differential Equations.* By A. R. FORSYTH. Third Edition. Pp. xvi., 511. 14s. 1903. (Macmillan.)
- Beginners' Algebra.* By M. S. DAVID. Pp. vi., 232. 2s. 6d. 1903. (Black.)
- Solid Geometry.* By F. HOCEVAR. (Translated and adapted by C. Godfrey and E. A. Price.) Pp. viii., 80. 1s. 6d. 1903. (Black.)
- A School Geometry.* Parts I. and II. By H. S. HALL and F. H. STEVENS. Pp. x., 140. 1s. 6d. 1903. (Macmillan.)
- Formulaire Mathématique.* By G. PEANO. Pp. xvi., 313-407. 1903. (Bocca, Turin.)
- Mathematics in the Elementary School.* By D. E. SMITH, and F. M. M'MURRY. Teachers College Record. March, 1903. Pp. 70. 30 c. (Macmillan Co.)
- The Monist.* Jan.—April, 1903. 2\$ per annum. (Kegan Paul.)
- Open Court.* Jan.—April, 1903. 1\$ per annum. (Kegan Paul.)

ERRATA, ETC.

Note [123. 1^a. 1. a.] Mr. Davis is informed that the method given on p. 262 is to be found in Johnston's *Analytical Geometry* (Macmillan), p. 271.

p. 213, lines 18, 19, 20, 22, and in first note read w for ω .

line 23 read $\mu\omega + \omega\rho$ for $\mu\omega + \omega\rho$.

line 25 read $\omega q_2 + \omega(\omega q_1 + q\rho - q_2\rho) + \mu\omega q$
for $\omega q_2 + \omega(\omega q_1 + q\rho - q_2\rho) + \mu\omega q$.

p. 263, line 23 read *prime* for *primer*.

MACMILLAN & CO.'S NEW GEOMETRIES to meet the New Requirements.

By H. S. HALL AND F. H. STEVENS.

A SCHOOL GEOMETRY. Parts I. and II. Part I.—LINES AND ANGLES, RECTILINEAL FIGURES. Part II.—AREAS OF RECTILINEAL FIGURES (containing the Substance of Euclid Book I.). By H. S. HALL, M.A., and F. H. STEVENS, M.A. Globe 8vo. 1s. 6d.

*** This work is based upon the recommendation of the
Mathematical Association.

PRACTICAL EXERCISES IN GEOMETRY. By W. D. EGGAR, M.A., Assistant Master at Eton College. Globe 8vo. 2s. 6d.

Royal College of Science Magazine.—"This is one of the most original books we have seen, and will be welcomed by advocates of the heuristic method of teaching, and indeed by all teachers who wish their pupils to be interested as well as instructed. . . . We give it our heartiest recommendation, and congratulate the author on its production."

THEORETICAL GEOMETRY FOR BEGINNERS.

By C. H. ALLCOCK, Senior Mathematical Master at Eton. Globe 8vo. 1s. 6d.
Engineer.—"It is well arranged, and should certainly prove useful to beginners in this branch of mathematics."

A NEW GEOMETRY FOR SCHOOLS. By S. BARNARD, M.A., and J. M. CHILD, B.A. Vol. I.—STRAIGHT LINE, CIRCLE, AREA. Crown 8vo. [Shortly.]

GEOMETRICAL DRAWING AND DESIGN. By J. HUMPHREY SPANTON. Adapted to the requirements of the Board of Education. Globe 8vo. 2s. 6d.

School World.—"Mr. Spanton's book thoroughly deserves a trial, and will, we think, prove an unqualified success."

PRACTICAL PLANE AND SOLID GEOMETRY FOR ELEMENTARY STUDENTS. By JOSEPH HARRISON, M.I.M.E., Assoc. M.Inst. C.E. Globe 8vo. 2s. 6d.

PRIMER OF GEOMETRY. Comprising the subject-matter of Euclid I.-IV. Treated by the methods of Pure Geometry. By H. W. CROOME SMITH, B.A. Globe 8vo. 2s.

Royal College of Science Magazine.—"Will speedily find favour, especially with teachers and students who have already found a want of common sense in the ordinary method of treating geometry."

A SHORT INTRODUCTION TO GRAPHICAL ALGEBRA. By H. S. HALL, M.A. Second edition. Revised and Enlarged. Globe 8vo. 1s.

Teachers' Aid.—"It is a capital introduction to this wide branch of algebraical research, and deserves to be extensively known."

NOW READY. WITH A CHAPTER ON GRAPHS.

ELEMENTARY ALGEBRA FOR SCHOOLS. By H. S. HALL, M.A., and S. R. KNIGHT, B.A. Globe 8vo. 3s. 6d. With Answers, 4s. 6d. Answers, 1s. KEY, for Teachers only, 8s. 6d.

THIRD EDITION NOW READY.

A TREATISE ON DIFFERENTIAL EQUATIONS. By ANDREW RUSSELL FORSYTH, Sc.D., LL.D., Math. D., F.R.S., Sadlerian Professor of Pure Mathematics. 8vo. 14s.

FIVE FIGURE LOGARITHMIC and other Tables. By ALEX. M'AULAY, M.A., Professor of Mathematics and Physics in the University of Tasmania. 16mo. 2s. 6d.

MACMILLAN & CO., LTD., ST. MARTIN'S STREET, LONDON, W.C.

NEW GEOMETRY

For Beginners, Theoretical and Practical.

By **RAWDON ROBERTS.**

Crown 8vo. Cloth, 1s. 6d.

MR. C. GODFREY, CHIEF MATHEMATICAL MASTER, WINCHESTER COLLEGE, in an address on "The Teaching of Mathematics," at the Annual Conference of the Association of Headmasters of Preparatory Schools, December 22nd, 1902, said:—
"As to Euclid or Euclid revised to meet the latest requirements, the only good book I have seen as yet is Roberts' 'New Geometry for Beginners' (*Blackie*)."

Educational Catalogue and Lists of New Mathematical
Books post free on application.

BLACKIE & SON, Ltd., 50 OLD BAILEY, LONDON, E.C.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament therunto."—BACON.

President :

Professor A. R. FORSYTH, LL.D., Sc.D., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.

R. LEVETT, M.A.

Prof. W. H. H. HUDSON, M.A.

Professor A. LODGE, M.A.

Professor G. M. MINCHIN, F.R.S.

Treasurer :

F. W. HILL, M.A., City of London School, London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.

H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

S. BARNARD, M.A.

R. F. DAVIS, M.A.

J. M. DYER, M.A.

C. GODFREY, M.A.

G. HEPPLE, M.A.

F. S. MACAULAY, M.A., D.Sc.

W. N. ROSEVEARE, M.A.

A. W. SIDMONS, M.A.

C. O. TUCKER, B.A.

A. E. WESTERN, M.A.

C. E. WILLIAMS, M.A.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "The Mathematical Gazette."

Store Hall

THE MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

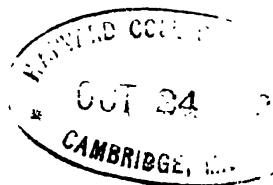
PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY



Vol. II., No. 41.

OCTOBER, 1903.

1s. 6d. Net.

The Mathematical Association.

THE ANNUAL MEETING of the MATHEMATICAL ASSOCIATION will be held in January, 1904, at King's College, Strand, W.C.

Notices of papers or discussions for the General Meeting in January should reach the Hon. Secretaries not later than December 1st.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

Problems and Solutions should be sent to
W. E. HARTLEY, 6 Benson Street, Cambridge.

All other correspondence concerning the Gazette should be addressed to
W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

Correspondence relative to the Mathematical Association should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, St. Paul's School, London, W.; H. D. ELLIS, 12 Gloucester Terrace, Hyde Park, W.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied, except on payment of 1s. per copy.

CONTENTS.

	PAGE
A CHAPTER ON ALGEBRA (<i>continued</i>). W. N. ROSEVEARE, M.A.,	325
THE SLIDE RULE AND ITS USE IN TEACHING LOGARITHMS. C. S. JACKSON, M.A.,	330
MATHEMATICAL NOTES. P. E. B. JOURDAIN, B.A.; R. F. MUIRHEAD, M.A.,	337
REVIEWS. R. F. DAVIS, M.A.; R. W. H. T. HUDSON, M.A.; C. S. JACKSON, M.A.; W. R. STRONG, F.I.A.,	342
QUERIES, SALE, AND EXCHANGE,	348
BOOKS, ETC., RECEIVED,	348

Forthcoming Articles and Reviews by:—

Prof. A. C. DIXON; Prof. T. J. I'a BROMWICH; Prof. A. LODGE; Prof. G. B. MATHEWS, F.R.S.; J. H. GRACE, M.A.; R. W. H. T. HUDSON, M.A.; E. M. LANGLEY, M.A.; W. P. WORKMAN, M.A.; H. W. RICHMOND, M.A.; A. N. WHITEHEAD, M.A.; E. W. BARNES, M.A.; F. S. MACAULAY, D.Sc., and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. No. 8 is out of print. A few numbers may be obtained from the Editor at the following prices, post free:

No. 7, 5s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 15s.

These will soon be out of print and very scarce.

THE MATHEMATICAL GAZETTE.

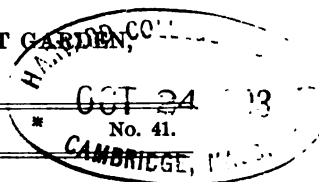
EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON:
GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. II.

OCTOBER, 1903.



A CHAPTER ON ALGEBRA.

(Continued from p. 306.)

- VII. The Exponential Expansion for Commensurable Indices.
- VIII. General Binomial Expansion.
- IX. On Incommensurable Indices.
- X. On the same Theorems for a Vector Variable.

VII.

The Exponential Expansion for Commensurable Indices.

If n is positive and commensurable, and $x^{\frac{1}{J}} = 1 + \frac{X}{J}$,

$$\begin{aligned} x^n &\equiv \left(1 + \frac{X}{J}\right)^I \\ &\equiv \sum_0^I \frac{I(I-1)\dots(I-i+1)}{[i]} \left(\frac{X}{J}\right)^i \\ &\equiv \sum_0^I \frac{(nX)^i}{[i]} \cdot 1 \left(1 - \frac{1}{I}\right) \left(1 - \frac{2}{I}\right) \dots \left(1 - \frac{i-1}{I}\right). \end{aligned}$$

Consider the coefficient of $\frac{(nX)^i}{[i]}$.

(1) It is less than unity.

(2) It can be written $1 \left(1 - \frac{i-1}{I}\right) \left(1 - \frac{1}{I}\right) \left(1 - \frac{i-2}{I}\right) \dots$

The sum of each pair of factors is the same, viz. $\left(2 - \frac{i-1}{I}\right)$;

$\therefore$ by § I., the product of the outermost pair $1 \left(1 - \frac{i-1}{I}\right)$ is least;

$\therefore$ the coefficient is greater than $\left(1 - \frac{i-1}{I}\right)^{\frac{1}{2}}$,

which, by § III., is greater than $1 - \frac{i(i-1)}{2I}$.

Therefore, if X is positive,

$$x^n \text{ lies between } \sum_0^I \frac{n^i X^i}{i!} \text{ and } \sum_0^I \left\{ 1 - \frac{i(i-1)}{2I} \right\} \frac{n^i X^i}{i!}$$

This lower limit $\equiv \sum_0^I \frac{n^i X^i}{i!} - \frac{n^2 X^2}{2I} \sum_0^I \frac{n^i X^i}{i!},$

which $> \left(1 - \frac{n^2 X^2}{2I} \right) \sum_0^I \frac{n^i X^i}{i!}.$

$$\therefore x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{i!} \text{ by less than } \frac{n^2 X^2}{2I} \sum_0^I \frac{n^i X^i}{i!};$$

but, since $x^n > \left(1 - \frac{n^2 X^2}{2I} \right) \sum_0^I \frac{n^i X^i}{i!}, \therefore \sum_0^I \frac{n^i X^i}{i!} < \frac{x^n}{1 - \frac{n^2 X^2}{2I}};$

$$\therefore x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{i!} \text{ by less than } \frac{\frac{n^2 X^2}{2I}}{1 - \frac{n^2 X^2}{2I}} \cdot x^n.$$

Now let I and J increase indefinitely, their ratio remaining n .

Then X takes the finite value $\log x$ and the above error becomes continuously smaller and ultimately vanishes.

Therefore, when X is positive, i.e. when $x > 1$,

$$x^n \equiv \sum_0^\infty \frac{(n \log x)^i}{i!}.$$

If X is negative, i.e. if $x < 1$, the terms of

$$\sum \frac{(nX)^i}{i!} 1 \left(1 - \frac{1}{I} \right) \dots \left(1 - \frac{i-1}{I} \right)$$

are alternately + and -; but it is still true that the series differs from

$$\sum \frac{n^i X^i}{i!} \text{ by less than } \sum \frac{i(i-1)}{2I} \frac{(-nX)^i}{i!}$$

where all the terms are positive;

$$\therefore x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{i!} \text{ by less than } \frac{n^2 X^2}{2I} \sum_0^I \frac{n^i (-X)^i}{i!}$$

Now let

$$x^{\frac{1}{J}} \equiv 1 + \frac{-X}{J};$$

then by the former work,

$$\sum_0^I \frac{n^i (-X)^i}{i!} < \frac{x^n}{1 - \frac{n^2 X^2}{2I}}$$

But $x^{\frac{1}{J}} x^{\frac{1}{J}} \equiv 1 - \frac{X^2}{J^2}$ which < 1 ;

$$\therefore x' < \frac{1}{x};$$

$$\therefore \text{the error} < \frac{\frac{n^2 X^2}{2I}}{1 - \frac{n^2 X^2}{2I}} \frac{1}{x^n}.$$

Hence, for all positive commensurable values of n , whether I, J are finite or not,

$$x^n \text{ differs from } \sum_0^I \frac{n^i X^i}{i!}, \text{ where } x^{\frac{1}{J}} \equiv 1 + \frac{X}{J},$$

by a quantity less than

$$\frac{\frac{n^2 X^2}{2I}}{1 - \frac{n^2 X^2}{2I}} x^{\pm n},$$

the upper sign being used if $x > 1$ and the lower if $x < 1$.

If J is increased indefinitely, this error vanishes, and, for all positive values of x ,

$$* x^n = \sum \frac{(n \log x)^i}{i!} \text{ when } n \text{ is positive.}$$

The case of a negative index presents no difficulty ;

for

$$x^{-n} = \left(\frac{1}{x}\right)^n = \sum \frac{\left(n \log \frac{1}{x}\right)^i}{i!} \text{ which } = \sum \frac{(-n \log x)^i}{i!}.$$

VIII.

The Binomial Expansion.

Let B_n stand for $x^n - \sum_0^{\kappa} (n)_i (x-1)^i$, where n has any value, and $(n)_i$ represents

$$\frac{n(n-1)\dots(n-i+1)}{1.2\dots i}.$$

The terms $(x-1)^i$ can be expanded in powers of x :

Assume that we so get

$$B_n \equiv x^n - [N_0 + N_1 x + N_2 x^2 \dots + N_{\kappa} x^{\kappa}],$$

where $N_0, N_1, \dots$ are algebraic functions of n of dimensions not greater than κ .

(1) If n is a positive integer less than κ (ν say), $B_n = 0$;

$\therefore N_0, N_1, \dots$ all vanish when $n = \nu$, except N_{ν} which = 1.

Hence

$$N_{\nu} = \frac{n(n-1)\dots(n-\kappa)}{n-\nu} \cdot c_{\nu},$$

where c_{ν} contains no n because the dimensions of the other factor are κ ;

To find c_{ν} we have $N_{\nu} = 1$ when $n = \nu$;

$$\therefore c_{\nu} = \frac{1}{\{\nu. \nu-1 \dots 1\} \cdot \{(-)^{\kappa-\nu} . 1.2 \dots \kappa-\nu\}} \text{ or } \frac{(-)^{\kappa-\nu}}{\kappa} (\kappa)_{\nu}.$$

(2) Again, B_n vanishes for all values of n if $x = 1$;

$$\therefore 0 = 1 - [N_0 + N_1 + \dots + N_{\kappa}].$$

Hence

$$B_n \equiv N_0(x^n - 1) + N_1(x^n - x) \dots + N_{\kappa}(x^n - x^{\kappa}),$$

which by the above may be written $n(n-1)\dots(n-\kappa) \sum_{i=0}^{\kappa} c_i \cdot \frac{x^n - x^i}{n-i}$.

* This expansion might have been more simply established by first confining ourselves to the case $n=1$ (i.e. $I=J$ in the above work), and afterwards using the relation $\log x^n = n \log x$. The less simple proof has been given for the sake of the result when I and J are not indefinitely increased.

(3) To obtain limits for the value of $\sum_{i=0}^{i=\kappa} c_i \frac{x^n - x^i}{n-i}$, we use the Exponential Expansion established in § VII.

We have
$$x^n \equiv 1 + \frac{nX}{1} + \frac{n^2 X^2}{2} + \dots$$

$$\equiv 1 + nX \left[1 + \frac{1}{2} \cdot \frac{nX}{1} + \frac{1}{3} \cdot \frac{n^2 X^2}{2} + \dots \right].$$

Now by § V., $\frac{1}{i} = \text{Av.} \left| \theta^{i-1} \right|;$

$$\therefore x^n \equiv 1 + nX \cdot \text{Av.} \left[1 + \theta nX + \theta^2 \frac{n^2 X^2}{2} + \dots \right]$$

$$\equiv 1 + nX \text{Av.} (x^{n\theta});$$

$$\therefore \frac{x^n - x^i}{n-i} \equiv x^n \cdot \frac{x^{i-n} - 1}{i-n} \equiv x^n \cdot X \cdot \text{Av.} \left| x^{(i-n)\theta} \right|;$$

$$\therefore B_n = \frac{n \dots (n-\kappa)}{\kappa} \cdot x^n X \text{Av.} \left| x^{-n\theta} \sum_{i=0}^{\kappa} (-)^{i-1} (\kappa)_i x^{i\theta} \right|$$

$$\equiv n \cdot (n-1)_\kappa X \text{Av.} \left| x^{n\theta} (x^\theta - 1)^\kappa \right|, \text{ where } \theta \equiv 1 - \theta.$$

Now
$$(x^\theta - 1)^\kappa < \frac{(x-1)^\kappa}{0};$$

$$\therefore B_n < n(n-1)_\kappa \left| \frac{(x-1)^\kappa}{0} \right| \cdot X \text{Av.} x^{n\theta},$$

and
$$X \text{Av.} x^{n\theta} \equiv \frac{x^n - 1}{n}, \text{ as before.}$$

Hence
$$x^n \equiv \sum_0^\kappa (n)_i (x-1)^i + \text{an error } (n-1)_\kappa (x-1)^\kappa \left| \frac{x^n - 1}{0} \right|.$$

This error vanishes as κ increases if $(x-1) < 1$ numerically.

In the special case of $x-1=1$ or $x=2$, the error vanishes ultimately if $(n-1)_\kappa$ vanishes.

Now the part of $(n-1)_\kappa$ after the signs of the factors have all become negative varies as $\left(1 - \frac{n}{\nu}\right) \left(1 - \frac{n}{\nu+1}\right) \dots \left(1 - \frac{n}{\kappa}\right)$, which, by § III., if $n > 0$

$$< \left(1 + \frac{1}{\nu}\right)^{-n} \left(1 + \frac{1}{\nu+1}\right)^{-n} \dots \left(1 + \frac{1}{\kappa}\right)^{-n} < \left(\frac{\nu}{\kappa+1}\right)^n;$$

and if $n = -f$, where $f < 1$,

$$\left(1 + \frac{f}{\nu}\right) \dots \left(1 + \frac{f}{\kappa}\right) > \left(1 + \frac{1}{\nu}\right)^f \dots \left(1 + \frac{1}{\kappa}\right)^f > \left(\frac{\kappa+1}{\nu}\right)^f;$$

$\therefore$ when $n > 0$ the error vanishes ultimately, but when $n < 0$ it increases indefinitely.

For the special case of $x-1 = -1$ or $x=0$ we must return to the beginning of the article.

We need only consider the case of $n > 0$, for otherwise x^n is not a finite quantity.

Thus, in this case, $B_n \equiv -N_0 = \frac{(n-1) \dots (n-\kappa)}{\kappa} (-)^{\kappa+1}$, since $(\kappa)_0 = 1$.

This vanishes ultimately, as before, if $n > 0$.

IX.

On Incommensurable Indices.

The condition of obedience to the law $a^m \times a^n = a^{m+n}$ establishes the meaning of a^n when n is any commensurable, but leaves it undefined when n is incommensurable. Indicating the incommensurable by a Greek letter, we now seek to give a meaning to x^ν . Our freedom in giving such a meaning is limited only by the condition of continuity, viz. that if ν lies between two commensurables n and n' , a^ν must be so defined that it lies between a^n and $a^{n'}$, whose meaning is already fixed.

Now we may suppose n to be $\frac{I}{J}$ and n' to be $\frac{I+1}{J}$.

Then as in § IV., $x^n = \left(1 + \frac{nX}{I}\right)^I$ and $x^{n'} = \left(1 + \frac{n'X}{I+1}\right)^{I+1}$,

where X is given by $x^{\frac{1}{J}} = 1 + \frac{X}{J}$.

Now consider the expression $\left(1 + \frac{\nu X}{I}\right)^I$.

(1) If X is positive,

since $\nu > n$, this $> \left(1 + \frac{nX}{I}\right)^I$ and $\geq \left(1 + \frac{n'X}{I+1}\right)^{I+1}$

according as $\left(1 + \frac{\nu X}{I}\right)^{\frac{I}{I+1}} \geq 1 + \frac{n'X}{I+1}$.

But by § III.,

$\left(1 + \frac{\nu X}{I}\right)^{\frac{I}{I+1}} < 1 + \frac{\nu X}{I+1}$ and therefore $< 1 + \frac{n'X}{I+1}$ since $\nu < n'$.

Therefore when $x > 1$, $\left(1 + \frac{\nu X}{I}\right)^I$ is a sound definition of x^ν , provided I and J are considered indefinitely great, and therefore X is $\log x$.

(2) If X is negative,

$$\left(1 + \frac{\nu X}{I}\right)^I < \left(1 + \frac{nX}{I}\right)^I$$

and $\left(1 + \frac{\nu X}{I}\right)^I > \left(1 + \frac{n'X}{I+1}\right)^{I+1}$, if $1 + \frac{\nu X}{I} > \left(1 + \frac{n'X}{I+1}\right)\left(1 + \frac{n'X}{I+1}\right)^{\frac{1}{I}}$;

now $\left(1 + \frac{n'X}{I+1}\right)^{\frac{1}{I}} < 1 + \frac{n'X}{I(I+1)}$;

$\therefore$ the last condition is satisfied if $1 + \frac{\nu X}{I} > \left(1 + \frac{n'X}{I+1}\right)\left(1 + \frac{n'X}{I(I+1)}\right)$.

Remembering that X is negative,

this becomes $\nu < n' + \frac{n'^2 X}{(I+1)^2}$, i.e. $< n' + \frac{X}{J^2}$,

which is true if J is large enough.

Hence in this case also, the definition $x^\nu \equiv \left(1 + \frac{\nu X}{I}\right)^I$, where I and J are taken indefinitely great, is valid.

Hence, reasoning as in § VII., we may say that

$$x^n \equiv 1 + n \log x + \frac{(n \log x)^2}{2} + \dots$$

for all values of n , commensurable or not.

It is easy to show that x^r as so defined, satisfies the laws of indices.

$$\text{Thus } x^\mu \cdot x^r \equiv \left(1 + \frac{\mu X}{I}\right)' \left(1 + \frac{r X}{I}\right)' \equiv \left(1 + \frac{\mu + r \cdot X}{I}\right)' \left[1 + \frac{\mu r X^2}{I^2 + \mu + r \cdot X}\right]' \\ = x^{\mu+r} \left[1 + \frac{a}{I^2 + bI}\right]' \text{ say ;}$$

and, by § III., $\left(1 + \frac{a}{I^2 + bI}\right)' > 1 + \frac{a}{I+b}$ and $< \frac{1}{1 - \frac{a}{I+b}}$;

and $\therefore (I \text{ being indefinitely great}) = 1.$

$$\text{Again, } \log x^\mu = \text{Lt } I \left[(x^\mu)^{\frac{1}{I}} - 1 \right] = \text{Lt } I \left[\left(1 + \frac{\mu X}{I}\right) - 1 \right] = \text{Lt } \mu X = \mu \log x ;$$

$$\therefore (x^\mu)^r \equiv \left(1 + \frac{r \log x^\mu}{I}\right)' \equiv \left(1 + \frac{\mu r \log x}{I}\right)' \equiv x^{\mu r} \text{ by definition.}$$

Thus in the Exponential Expansion the number e can now be introduced : and we can prove $x = e^{\log x}$.

X.

On the Same Theorems for a Vector Variable.

The work in § VII. is applicable to vector values of x . The logarithm being defined exactly as before, takes the form $(\log a + \sqrt{-1} \cdot \alpha)$, where a , α are the length and inclination of x . In the expression for the error, X and x are replaced by their moduli, and the error vanishes ultimately. The Binomial Theorem follows by § VIII. (provided that x is within two right angles of the line denoted by 1), the words 'lies between' at the end assuming a suitable meaning for vectors. The logarithmic expansion can be deduced from the exponential by expanding n^i in the finite algebraic form $\sum_{j=1}^{j=i} (n)_j \cdot i_j$, whence

$$(x-1)^i \equiv \sum_{j=1}^{j=i} \frac{X^j}{j!} \cdot i_j \quad \left| \text{with an error vanishing ultimately as } \kappa \text{ is increased.} \right.$$

Certain simple properties of the coefficient i_j establish the logarithmic expansion and other less simple expansions for higher powers of $\log x$.

HARROW, July, 1903.

W. N. ROSEVEARE.

THE SLIDE RULE AND ITS USE IN TEACHING LOGARITHMS.

1. I should like at the outset to disclaim any intention of laying down the law dogmatically. The opinions expressed are stated definitely, but they are submitted with deference for your consideration.

My object is to argue two points :

- (1) That the construction of a simple form of slide rule furnishes beginners with a good mode of approaching the subject of logarithms.
- (2) That the use of the slide rule at any earlier stage than has been customary deserves every encouragement.

2. It is a commonplace remark that a principle may be presented to a pupil in such an abstract form that his mind fails to assimilate it. This

maxim is, however, not always respected. Galileo, after years of experimenting, reached clear notions about acceleration. There were great men before Galileo who never attained clearness on this point. Yet our textbooks polish off acceleration in a few paragraphs.

Even the Heuristic method, though it is in substance a protest against this form of error, is not, at any rate as presented by some of its advocates, entirely free from it. For example, a boy is invited, after contemplating a lump of coal for five minutes, to formulate a statement. The suggestion is that he will be moved to declare that "matter cannot set itself in motion." That is, the ordinary boy is to arrive in five minutes at a statement, the precise formulation of which is an item in the list of Newton's titles to fame.

These authorities—with sanguine ideas as to the ability of the average pupil—not unnaturally adopt as their cardinal rule "never teach anyone anything," an odd principle for a teacher. The better maxim is never teach a pupil that which he can find out for himself with a reasonable expenditure of time, patience and pains.

This is the true rule, because the object of education is to enable people to find out things for themselves. The application of this guiding rule is not easy; it is apt to be misunderstood and resented by pupils who have been brought up on the plan of being shewn everything; it does not pay in examinations, it results in less ground being covered; and it requires an amount of time, effort, and individual attention from the teacher which it is often, under existing conditions, simply impossible for him to give, but we all now see that it should be our ideal.

But if the plan of making the pupil cut some of the steps for himself is to be followed, the ascent, for the great majority of pupils, must be a very easy one and must consequently appear somewhat long. In the end, however, these slow experimental methods will prove to surpass purely didactic teaching in educating as distinguished from information-inserting.

3. I suggest that it is possible to approach the thorny subject of logarithms in an experimental fashion with advantage. I call the subject of logarithms a thorny one because it is within my personal knowledge that to many successful Woolwich candidates calculation by logarithms is an occult process, savouring of the black art.

How many times I have been asked how it is that

$$\log(a+b) \text{ is not equal to } \log a + \log b,$$

and wondered whether a zealous attempt to explain has had any effect when the many previous zealous and skilful explanations which candidates received at school have obviously had none. This fact suggests that the subject might perhaps be approached with advantage in a different manner.

For instance, let us begin by trying to construct a machine to do addition and subtraction. After a discussion, in which the teacher will have to be the predominant partner, we arrive at something like this: each boy using a pair of paper slips made by himself, in which a step of, say, an inch along the uniformly divided scale brings us to a number which is 2 more than the one we started with.

			0	1	2	3	4	5	6	7	8	9	10
0	1	2	3	4	5	6	7	8	9	10			
A			P					B					

If the strips * are shown overlapping as above we see that they mechanically

* A couple of wooden slips with figures stencilled on them were used for this and subsequent illustrations. Better than stencils are the indiarubber figures and letters sold for printing shop window tickets. I am indebted to Mr. J. H. Naylor, R.M. Academy, for this 'tip,' most useful when diagrams are often wanted.

add 3 to 5 ; for instance, for $AP + PB = AB$. The same setting mechanically subtracts 5 from 8.

Stated in a slightly different way. . . . For any 'setting' or relative position of the strips the DIFFERENCE between any corresponding pair of numbers on the scales is constant.

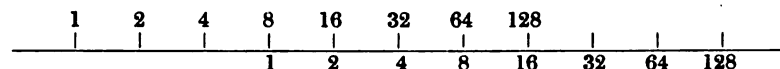
4. We next suggest the problem of making a machine for multiplication, and arrive at the idea that, as in the addition machine, a step of say an inch to the right *adds* 2 we will try whether we cannot make a machine in which a step of an inch to the right multiplies by 2.



This leads to a uniformly divided scale graduated as shewn.

We are now able to work out, using a pair of such scales, certain arithmetical problems—in fact we have a rudimentary slide rule.

Set the strips overlapping—thus, for instance : [I need hardly say that of course each pupil should make his own model with two strips of paper].



Notice that any number on the upper strip is 8 times that just below it, and that hence we can, from this particular 'setting' or relative position of the strips, read off the answers to certain sums in arithmetic thus :

MULTIPLICATION.—The fact that 8 times 4 is 32 is shown thus : $\begin{smallmatrix} 8 & 32 \\ 1 & 4 \end{smallmatrix}$.

DIVISION.—The fact that 32 divided by 4 is 8 is shown in the same way.

PROPORTION.—The fact that 32 is to 4 as 128 is to 16 is shown thus :

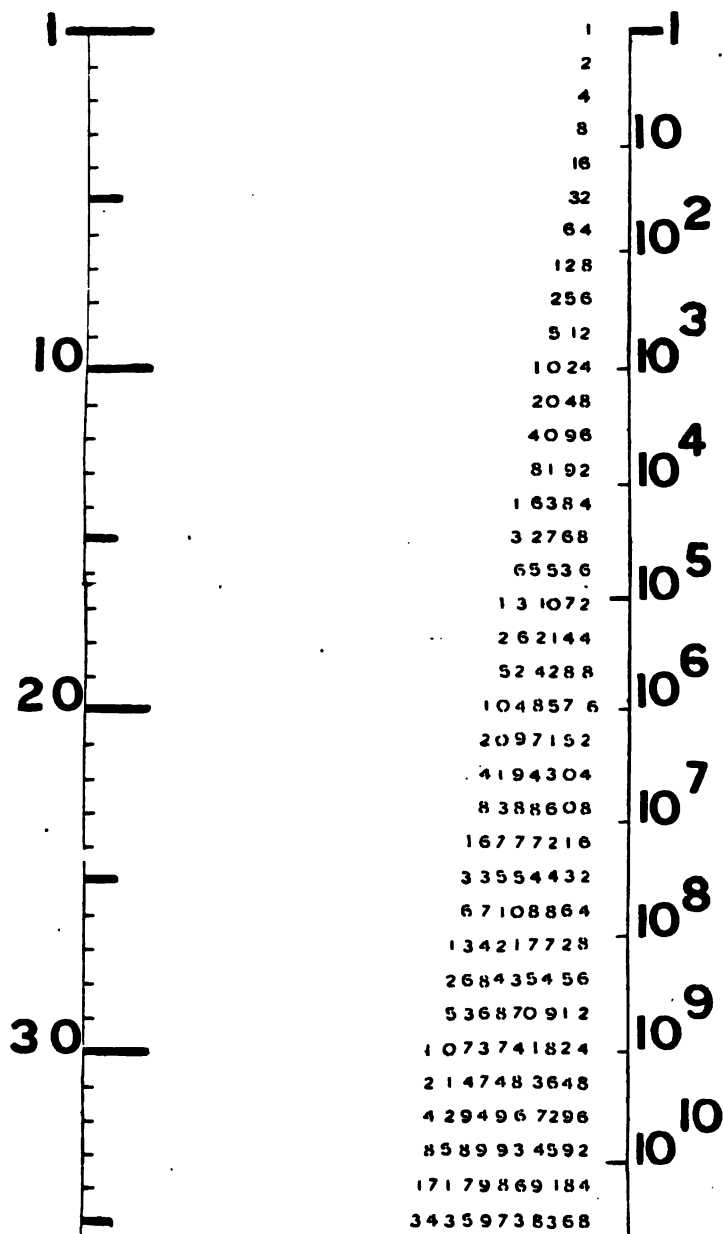
$\begin{smallmatrix} 32 & 128 \\ 4 & 16 \end{smallmatrix}$ The scales may, of course, be extended to the left if desired.

Other sums may now be tried in a similar way, but with the strips of paper set to a different overlap. We observe that in this case—For any 'setting' or relative position of the strips the **RATIO** of any corresponding numbers* on the scales is constant.

It is evident that in its present form the instrument is not a practical one, for only a few numbers can be dealt with. We could clearly construct another instrument to deal with the numbers 3, 9, 27, 81, etc., but this does not overcome the difficulty. We made our scales on the supposition that a step of an inch to the right effected multiplication by 2. It is clear that a step of 2 inches to the right multiplies by 4, of 3 inches to the right multiplies by 8, and so on, and we infer that any step of definite amount corresponds to multiplication by some definite number. The problem then is reduced to finding the proper position for any required number on the scales.

We have now got to the pinch of the case. A little time given to the class is certain to produce some good and some poor suggestions. For example, you are certain to be told that 3 should come half way between 2 and 4. The line to follow will depend on the suggestions received—so the following is only one of many possible modes of advance. Let us consider diagram (i) supposed, to begin with, to contain the marks for 10, 100, 1000, 10000 and so on. These marks will, as we have seen, be all equidistant, say a decimetre apart. Now, looking at the diagram which is at present a

* Note that if we invert one of the adding scales we get the **SUM** of corresponding numbers constant, and if we invert one of the multiplying scales we get the **PRODUCT** of corresponding numbers constant.



multiplying machine with 10, 100, and so on marked on it, we want to know where to put the mark for 2. Suppose the distance from 1 to 2 ought to be x decimetres, we have to find x . Evidently the distances 2-4, 4-8, 8-16, . . . should each be equal to x . Now if we take any value for x and mark in the series of numbers as shewn in the diagram, it is clear that all the numbers shewn, both the multiples of 10 and the multiples of 2 should always come in order. A larger number should never be reached before a smaller one. Hence we must reach 1000, the mark for which is 3 decimetres from A , before we reach 1024, the mark for which should be $10x$ decimetres from A .

Hence
$$x > \frac{3}{10} > \cdot 3.$$

Again we must reach the point which should be $33x$ decimetres from A before we reach the point which is 10 decimetres from A .

Hence
$$x < \frac{10}{33} < \cdot 303.$$

Inviting attempts to find x still more closely, we soon find that

$$103x > 31, \quad x > \cdot 30097,$$

$$93x < 28, \quad x < \cdot 30107;$$

or, if it is considered worth while* to go further,

$$196x > 59, \quad x > \cdot 3010204,$$

$$485x < 146, \quad x < \cdot 3010309.$$

Of course all this is nothing but $2^{103} > 10^{31}$, and so on, which, however, is in too abstract a form for beginners.

$$\frac{1}{\leftarrow \log n \rightarrow} n$$

$$\frac{1}{\leftarrow \log f(x) \rightarrow} x$$

$$\frac{m \leftarrow \log n - \log m \rightarrow n}{x \leftarrow \log f(y) - \log f(x) \rightarrow y}$$

We may now give x its technical name, logarithm of 2, and the calculation of the four figure logarithms of 3, 4, 5, 6, 7, 8, 9, might be undertaken—affording a lesson in the art of saving labour by thinking, for only $\log 3$ and $\log 7$ require any fresh calculations—and even these can be found from the diagram by a little ingenuity. It is my firm conviction that a great part of the difficulty which boys find in logarithms is due to the fact that the mode of calculating logarithms is to them a mystery. If they are once shewn how any logarithm can be calculated, they will, I think, make no difficulty in accepting the fact that the work has been done, once for all, by

* Shanks's table of powers of 2 is fairly accessible.

others. We are now in a position to construct a primitive slide rule and to work out a number of questions on it.

5. I now come to the utility of the slide rule. A distinguished French mathematician has spoken of "that inexplicable prejudice against the slide rule, against which it is impossible to protest too strongly." This prejudice is dying out, but it exists. The Diagram (2) shews the mathematical principle of the ordinary slide rule so far as it goes beyond what we have already discussed. You will observe that if the scales 1 and 2 are placed in conjunction with their units coinciding we must have $f(x)=n$, so that two such scales furnish us with a table of 'natural' values of $f(x)$. The diagram (2) also shews that we have the means of multiplying or dividing by $f(x)$, in fact of working the equation

$$\log n - \log m = \log f(y) - \log f(x), \text{ or } \frac{m}{f(x)} = \frac{n}{f(y)}.$$

It is clear that with such an instrument we can work out any question which can be done with logarithms; and several which can not. The slide rules in the market usually contain two scales for $\log n$, one on the ruler and one on the slide and logarithmic scales for some of the following functions:

$f(x)=x^2, x^{-1}, \sin x$ (which of course serves also for $\cos x$), $\tan x$, $\log x$, and a plain scale.

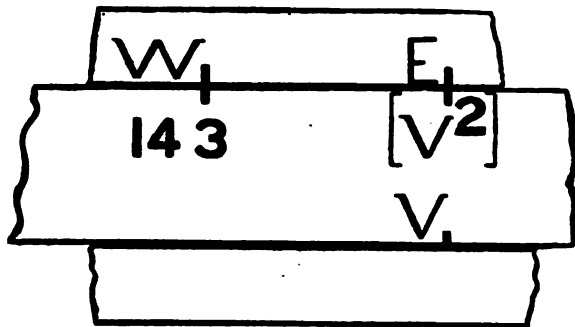
The $\frac{1}{x}$ scale is not very much needed, for we obtain it at once by putting the ordinary slide upside down.

Now, I claim for the use of the slide rule many advantages.

First. In very many cases an isolated numerical example does not bring out the essence of a formula. We want to observe the effect on the result of changes in the data. With a slide rule we can work out fifty numerical examples quicker than the answers can be written down.

The K.E. of a shell weighing W lbs. moving at V f.s. is

$$\frac{WV^2}{2g \times 2240} \text{ ft. tons} = \frac{WV^2}{143000} \text{ ft. tons.}$$



The diagram (3) shows how, taking an initial velocity of 2700 f.s. and a given weight of shell, we can read off the remaining K.E. E at any remaining velocity V . People who do not think say that the slide rule in these cases prevents a boy from thinking. We must distinguish. When I am trying to teach a boy about energy I don't want him to be thinking of multiplication and division; I welcome any means of preventing him from thinking about multiplication and division. I admit that the slide rule economises useless thought.* I consider this a merit.

* Mach., *Science of Mechanics*, p. 488, urges this point very forcibly.

6. Second, the slide rule familiarises us with the values of the trigonometrical functions, and enables *really accurate* graphs to be plotted very quickly. For instance, diagrams of Fourier series* can be rapidly and accurately drawn showing each term and the result of compounding any number of terms.

I have, we all have, again and again been told that there is no angle whose tangent is 3, no angle whose sine is $\frac{1}{2}$.

Mr. Siddons spoke in January of the undue importance which has been attached to 18°, 30°, 45° as compared with other angles equally worthy of respect. But that is not the worst: a gentleman has actually printed the formula, for boys to learn,

$$\sin 20\frac{1}{2}^\circ = \frac{\sqrt{3}}{5} \text{ nearly.}$$

We have all seen a good many queer formulæ, but I confess I have never seen anything to equal this. The use of a slide rule renders such nonsense impossible.

The third advantage I claim is certainty. In real life you generally needn't do a question by the shortest way, nor very quickly; but it must be right. With a slide rule there is no excuse for not checking your work.

Cube roots, plane and spherical triangles, quadratic and cubic equations, can with a little management be worked out independently with a slide rule, but at any rate they can be very easily checked by it.

Fourthly, how many authorities have denounced the practice of students who assert that the circumference of a penny bun is

$$9\cdot734933762749 \text{ inches,}$$

or that the angle of inclination of a shaky chimney-pot is 84° 17' 34". Yet how invariably boys make this mistake,† and how difficult it is to persuade them that it is a mistake. They call it accuracy.

If a class work out the value of $\sqrt{m \cdot n}$,

- (i) graphically on a small and large scale;
- (ii) with a slide rule;
- (iii) with 4 figure logarithms;
- (iv) with 7 figure logarithms;
- (v) by actual multiplication and extraction of the square root;

they cannot very well help seeing that each method has its own degree of accuracy, and that to give results which the method does not warrant is to tell an untruth. By supposing m and n only approximately known, we can see when the inevitable error in the result due to the imperfections of the data is appreciably added to by the approximate method of calculation adopted. I can testify that such work undertaken by a class arouses considerable interest; and that the necessary logarithmic work is done without much feeling that it is against the collar.

It is noteworthy that the exact error (due to the slide rule) committed in working any problem can always be found. This familiarises pupils with some of the leading results in the mathematical theory of errors; for instance, as has been already implied—that, to do justice to data, the Probable Error due to the use of approximate methods of calculation should not exceed about $\frac{1}{2}$ of the P.E. due to the imperfections of the data.

* Such diagrams are given in Byerley, "Fourier Series," p. 63; or Merriman & Woodward, *Higher Mathematics* (article by Professor Byerley), p. 199.

† Of course "It's near enough" is a conclusion pupils need no encouragement to adopt. But do not let us teach accuracy by inaccuracy. There are occasions when 7 or even 10 figure logarithms are called for; geodesic surveys furnish plenty of instances; but do not let us bring in 7 figure logarithms 'nisi dignus vindice nodus insiderit.'

Again, let us find the cube roots of the numbers 2, 3, 4, 5, 6, 7, 9...with a slide rule. Here we have an exact graduation to set the slide to; and we find, for instance, that the P.E. of the result is, say, a unit in the fourth significant figure. Now let us take the cube roots of 2.16, 3.16, 4.16... An additional source of error arises from the fact that we have to estimate, or interpolate, for the position of the 6. This causes the P.E. of the result to be perceptibly larger. Such examples, which may be multiplied almost indefinitely, give a real insight into the way in which a result is affected by several component sources of error.

There is perhaps no paramount scientific principle so often disregarded as this—always have some idea of the degree of accuracy of which your method is capable, and of the degree of accuracy of your own work.

To take a single case—our whole national system of competitive examination rests on a total disregard of this principle. This, however, is bordering upon topics of controversy.

I have one thing to say in conclusion. If anything in this paper is of value, the credit is largely due to Colonel H. C. Dunlop, R.A., with whom I have constantly discussed slide rule problems for some years.

The Logocyclic Slide Rule.—The special object of the logologarithmic slide is to enable the user to read off powers of numbers in the same way as the common slide rule enables us to read products.

Putting $f(x) = \log x$ in the principal equation, we have at once from the diagram $\log \log y - \log \log x = \log n - \log m$ or $y^m = x^n$, while, as special cases, if $m=1$, $y=x^n$, and if $x=10$, $n=\log_{10} y$. Thus powers and logarithms can be read off at a glance.

Numerous applications at once suggest themselves. As an instance of the use of the slide in getting out empirical formulæ, we may take Dr. Vincent's formula (which, perhaps, ought hardly to be termed empirical), viz.: atomic weight of the n^{th} element, when the elements are arranged in order of their atomic weights, is (in general) $(n+2)^{1.21}$.

It is a singular fact that the logolog scale has repeatedly been invented, forgotten, and reinvented. The first inventor was Dr. Roget. It was rediscovered and patented by Captain Thomson, R.A.; again, in Germany, by M. Blanc (Dyck's *Catalog*, p. 145), and again, almost simultaneously, reinvented by Professor Perry, and reintroduced by Colonel Dunlop and myself. The use of a $\log(-\log x)$ scale for numbers less than unity was one fondly thought new, but in this idea, which Professor Perry also brought forward, we were all anticipated by M. Blanc.

R. M. ACADEMY,
WOOLWICH.

C. S. JACKSON.

MATHEMATICAL NOTES.

128. [R. 6. b.] *On Gauss' Principle of Least Constraint and the Equations of Mechanics.*

The object of this note is to point out the remarkably simple principle to which Gauss' principle leads. The new principle is much simpler—not being an integral principle—than that of Hamilton,

$$\delta \int_0^t (T + U) dt = 0,$$

but, like it, is independent of coordinates, and leads at once to equations in generalised coordinates which have important properties.

1. Consider any material particle of mass m_r , whose coordinates are (x_r, y_r, z_r) at time t , and which forms part of a system. If m_r were quite

free and (X_v, Y_v, Z_v) were thus the only forces acting on it, its coordinates at time $t+dt$ would be

$$\left. \begin{aligned} x_v + \dot{x}_v dt + \frac{1}{2} \frac{X_v}{m_v} dt^2, \\ y_v + \dot{y}_v dt + \frac{1}{2} \frac{Y_v}{m_v} dt^2, \\ z_v + \dot{z}_v dt + \frac{1}{2} \frac{Z_v}{m_v} dt^2. \end{aligned} \right\} \dots\dots\dots(1)$$

However, owing to the connexions, the actual coordinates at the time $t+dt$ are

$$x_v + \dot{x}_v dt + \frac{1}{2} \ddot{x}_v dt^2. \dots\dots\dots(2)$$

The square of the difference between the positions (1) and (2) is then

$$\frac{dt^4}{4} \left\{ \left(\frac{X_v}{m_v} - \ddot{x}_v \right)^2 + \left(\frac{Y_v}{m_v} - \ddot{y}_v \right)^2 + \left(\frac{Z_v}{m_v} - \ddot{z}_v \right)^2 \right\}.$$

Now, Gauss' principle* states that the sum of these squared deviations, each multiplied by the proper mass, is a minimum. That is to say, the actual accelerations make the sum a minimum; or, for our purpose, the variation, which must affect only the accelerations and not the positions or velocities, must be zero. Thus, if δ denote such a variation, Gauss' principle becomes

$$\delta \sum_v \{ (X_v - m_v \ddot{x}_v)^2 + \dots \} = \sum_v \{ (X_v - m_v \ddot{x}_v) \delta \ddot{x}_v + \dots \} = 0. \dots\dots\dots(3)$$

There is an analogy with the variational form of d'Alembert's principle, but it must be remembered that here the δ 's are of different meaning. Here we have

$$\delta x_v = \delta y_v = \dots = \delta \dot{x}_v = \delta \dot{y}_v = \dots = 0,$$

but $\delta \ddot{x}_v, \dots$ are not zero; while, with d'Alembert's variations, $\delta x_v, \dots$ are not zero.

2. The equation (3) may be written

$$\begin{aligned} \sum_v (X_v \delta \ddot{x}_v + Y_v \delta \ddot{y}_v + Z_v \delta \ddot{z}_v) &= \sum_v m_v (\ddot{x}_v \delta \ddot{x}_v + \ddot{y}_v \delta \ddot{y}_v + \ddot{z}_v \delta \ddot{z}_v) \\ &= \frac{1}{2} \delta \sum_v m_v (\ddot{x}_v^2 + \ddot{y}_v^2 + \ddot{z}_v^2). \dots\dots\dots(4) \end{aligned}$$

If now we denote $\frac{1}{2} \sum_v m_v (\ddot{x}_v^2 + \ddot{y}_v^2 + \ddot{z}_v^2),$

which is independent of coordinates, by $\mathfrak{T}$, in analogy with the kinetic energy T ; and assume, for simplicity, a force-function $\mathfrak{U}$ to exist, so that

$$\delta \mathfrak{U} = \sum_v (X_v \delta \ddot{x}_v + \dots);$$

the equation (4) takes a form independent of coordinates,

$$\delta \mathfrak{U} = \delta \mathfrak{T} \quad \text{or} \quad \delta (\mathfrak{T} - \mathfrak{U}) = 0. \dots\dots\dots(5)$$

This form may now play the part of Hamilton's principle in the deduction of equations of motion in generalised coordinates, but the use of (5) is still simpler than the use of Hamilton's principle.

3. Let $q_1, q_2, \dots, q_n$ be the generalised coordinates of the system; then $\mathfrak{U}$ is a function of $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, \ddot{q}_1, \dots, \ddot{q}_n$, and $\mathfrak{T}$ of these and t , explicitly. Remembering that the δ affects only the accelerations, (5) gives

$$\sum_{v=1}^n \left\{ \frac{\partial \mathfrak{T}}{\partial \ddot{q}_v} \delta \ddot{q}_v - \frac{\partial \mathfrak{U}}{\partial \ddot{q}_v} \delta \ddot{q}_v \right\} = 0.$$

* "Ueber ein neues allgemeines Grundgesetz der Mechanik" (1829; *Werke*, v., 23-28).

This is equivalent to the system

$$\frac{\partial \mathcal{U}}{\partial q_\nu} - \frac{\partial \dot{\mathcal{U}}}{\partial \dot{q}_\nu} = 0. \quad (\nu = 1, 2, \dots n.)$$

We at once verify that $\frac{\partial \ddot{\mathcal{U}}}{\partial \ddot{q}_\nu} = \frac{\partial \mathcal{U}}{\partial q_\nu}$;

so that the equations of motion of a system have the simple form

$$\frac{\partial \mathcal{U}}{\partial q_\nu} = \frac{\partial \mathcal{U}}{\partial \ddot{q}_\nu}. \quad (\nu = 1, 2, \dots n.) \dots\dots\dots(6)$$

4. Mr. Whittaker remarked to me that the equations (6) had been already given by Appell. In fact, in 1899 and 1900, Appell published* several notes drawing attention to the remarkable fact that the equations (6)—which he obtained in quite a different way from mine—apply to the cases of non-holonomous systems (*i.e.* in which the equations of condition are non-integrable differential equations; these occur, *e.g.* in cases of rolling motion), in which Lagrange's equations fail. I shall show this in another manner, which appears to me not uninteresting. In fact, I shall show that the equations (6) are equivalent to an extended form of Lagrange's equations†—a form which reduces to the usual form when the system is holonomous.

5. Suppose the equations of condition are given in the non-integrable form

$$dx_r = \sum_{\nu=1}^n Q_{r,\nu} dq_\nu + Q dt, \dots; \dots\dots\dots(7)$$

so that $\dot{x}_r = \sum_{\nu=1}^n Q_{r,\nu} \dot{q}_\nu + Q$, ...; and $\frac{\partial \ddot{x}_r}{\partial \ddot{q}_\nu} = \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu}$.

If the equations (7) are integrable, and we can then assert that

$$Q_{r,\nu} = \frac{\partial x_r}{\partial q_\nu},$$

we also have

$$\frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} = \frac{\partial x_r}{\partial q_\nu}.$$

Then, putting

$$w_r^2 = \dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2,$$

$$\begin{aligned} \text{we have } \frac{\partial \mathcal{U}}{\partial \ddot{q}_\nu} &= \frac{1}{2} \sum_r m_r \left(\frac{\partial w_r^2}{\partial \dot{x}_r} \cdot \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} + \frac{\partial w_r^2}{\partial \dot{y}_r} \frac{\partial \dot{y}_r}{\partial \dot{q}_\nu} + \frac{\partial w_r^2}{\partial \dot{z}_r} \frac{\partial \dot{z}_r}{\partial \dot{q}_\nu} \right) \\ &= \sum_r m_r \left(\dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} + \dots \right) = \sum_r m_r \left\{ \frac{d}{dt} \left(\dot{x}_r \frac{\partial \dot{x}_r}{\partial \dot{q}_\nu} + \dots \right) - \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \ddot{q}_\nu} + \dots \right) \right\} \\ &= \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r \left(\dot{x}_r \frac{\partial \ddot{x}_r}{\partial \ddot{q}_\nu} + \dots \right), \end{aligned}$$

where $T = \frac{1}{2} \sum_r m_r (\dot{x}_r^2 + \dot{y}_r^2 + \dot{z}_r^2)$.

$$\text{Hence } \frac{\partial \mathcal{U}}{\partial \ddot{q}_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r (\dot{x}_r \dot{q}_{r,\nu} + \dot{y}_r \dot{q}_{r,\nu} + \dot{z}_r \dot{q}_{r,\nu}'');$$

and so Lagrange's equations, generalised for non-holonomous systems, are

$$\frac{\partial \mathcal{U}}{\partial q_\nu} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_\nu} \right) - \sum_r m_r (\dot{x}_r \dot{q}_{r,\nu} + \dots). \dots\dots\dots(8)$$

* *Comptes Rendus*, t. cxxix. (1899), pp. 317-320, 459-460; *Crelle's Journal*, Bd. cxxi. (1900), pp. 310-319; Bd. cxxii. (1900), pp. 205-209.

† Ferraers: "Extension of Lagrange's Equations," *Quart. Journ.*, vol. xii. (1873), pp. 1-5.

Now if, and only if, the conditions (7) are total differentials,

$$Q_{r,v} = \frac{\partial x_r}{\partial q_v}, \quad Q'_{r,v} = \frac{\partial y_r}{\partial q_v}, \quad Q''_{r,v} = \frac{\partial z_r}{\partial q_v};$$

and the last term of (8) reduces to

$$\frac{\partial T}{\partial \dot{q}_v},$$

and (8) takes the usual Lagrangian form.

The equations (6) are then equivalent to the system (8), and have the advantage of being expressed wholly in generalised coordinates; while Lagrange's equations,

$$\frac{\partial U}{\partial q_v} = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_v} \right) - \frac{\partial T}{\partial q_v}, \quad (v=1, 2, \dots, n)$$

result from (6) or (8) for the special case of holonomous systems.

PHILIP E. B. JOURDAIN, B.A., Trinity College, Cambridge.

129. [K. 20. a.] *Two Trigonometrical Notes.*

I. There is a large class of formulas by which one length (which may be supposed unknown) is expressed in terms of another length (which may be supposed known), and of the trigonometrical ratios of certain angles. Such formulas are often easily established by the aid of the artifice of expressing the ratio of the unknown to the known as a product of two or more ratios, each of which is either a trigonometrical ratio of a known angle or is the ratio between two sides of a triangle whose angles are known, and therefore expressible as the ratio between two known sines.

For example, take the problem: *To express the in-radius of a triangle in terms of one side and two angles.* With the usual notation we have to express r in terms, say, of a, B, C .

From I , the in-centre, let ID be drawn perpendicular to BC . We have

$$\begin{aligned} \frac{r}{a} &= \frac{ID}{BC} = \frac{ID}{IC} \cdot \frac{IC}{BC} = \sin ICD \times \frac{\sin IBC}{\sin BIC} \\ &= \sin \frac{C}{2} \times \frac{\sin \frac{B}{2}}{\sin \left(180^\circ - \frac{B}{2} - \frac{C}{2} \right)} = \frac{\sin \frac{C}{2} \sin \frac{B}{2}}{\sin \frac{B+C}{2}}. \end{aligned}$$

Hence the well-known result,

$$r = a \sin \frac{B}{2} \sin \frac{C}{2} / \cos \frac{A}{2}.$$

Again take this problem: *Given the distance between two points A, B on a horizontal plane, and the angles of elevation α, β of a point C in the vertical plane containing AB , as seen from A and from B respectively, to find the vertical height of C above the level of AB .*

Let CD be the perpendicular from C on AB produced.

$$\begin{aligned} \text{Then} \quad \frac{CD}{AB} &= \frac{CD}{CA} \cdot \frac{CA}{AB} = \sin CAD \times \frac{\sin CBA}{\sin ACB} \\ &= \sin \alpha \times \frac{\sin (180^\circ - \beta)}{\sin (\beta - \alpha)}; \end{aligned}$$

$$\therefore CD = AB \sin \alpha \sin \beta / \sin (\beta - \alpha).$$

It would be easy to multiply examples. These two form part of the usual bookwork of elementary trigonometry, but are often proved in the

text-books by longer methods, and without any clue by which the beginner could invent a proof for himself.

It will be seen that we have to choose one or more lengths in the figure as intermediaries between the known and the required lengths, in such a way that the ratio of unknown to known is expressible identically as compounded of known ratios.

II. A large proportion of the exercises usually set on heights and distances, not confined to one plane, are reducible to easy *plane* problems by projecting the figure on the horizontal plane, and expressing the horizontal lines in terms of some height or distance, given or required, and the trigonometrical ratios of given angles of elevation. An easy example will suffice to indicate the method.

From a station *A*, a balloon is observed at *P* at an elevation of θ° , its bearing being *N*. The balloon is known to be moving horizontally at the rate of 6 miles per hour. After $\frac{1}{2}$ hour it is seen from *A* at a point *Q* in direction *N.W.* at an elevation of ϕ° . Find *h*, the height of the balloon in miles.

Let *B* and *C* be the 'horizontal projections' of *P* and *Q*. Then

$$AB = h \cot \theta^\circ, \quad AC = h \cot \phi^\circ, \quad BC = PQ = 1\frac{1}{2} \text{ miles, and } \hat{BAC} = 45^\circ.$$

In the triangle *BAC* we have to relate the elements *AB*, *BC*, *CA*, $\hat{BAC}$, which is done by the formula

$$BC^2 = AB^2 + AC^2 - 2AB \cdot AC \cos BAC.$$

$$\text{This gives} \quad \frac{9}{4} = h^2 \cot^2 \theta^\circ + h^2 \cot^2 \phi^\circ - h^2 \cot \theta^\circ \cot \phi^\circ \sqrt{2}.$$

Hence *h* in terms of the data.

R. F. MUIRHEAD.

130. [A. 1. b.] Note on Mr. Roseveare's "Chapter on Algebra."

In Section III. inequalities equivalent to those in Chrystal's *Algebra*, Chapter XXIV., § 7, are deduced in an interesting way from the results of Section II.; but perhaps the proof given in 'Chrystal,' depending on the inequality $(p-q)\{q(x^p-1)-p(x^q-1)\} \leq 0$ (*x* being positive), is more natural. The most troublesome part of Chrystal's procedure, namely, the proof of the last-mentioned equality, may be simplified by substituting the following proof, the idea of which was suggested to me some time ago by a friend.

If *x* is positive, and *i* any positive integer,

$$\begin{aligned} & i(x^{i+1}-1) - (i+1)(x^i-1) \\ &= (x-1)\{i(x^i+x^{i-1}+x^{i-2}+\dots+x+1) - (i+1)(x^{i-1}+x^{i-2}+\dots+x+1)\} \\ &= (x-1)\{ix^i - x^{i-1} - x^{i-2} - \dots - x - 1\} \\ &= (x-1)\{(x^i - x^{i-1}) + (x^i - x^{i-2}) + \dots + (x^i - x) + (x^i - 1)\} \\ &= (x-1)^2\{x^{i-1} + x^{i-2}(x+1) + x^{i-3}(x^2+x+1) + \dots \\ &\quad + (x^{i-1} + x^{i-2} + \dots + x + 1)\} \\ &= (x-1)^2\{ix^{i-1} + (i-1)x^{i-2} + (i-2)x^{i-3} + \dots + 2x + 1\}, \end{aligned}$$

which is ≤ 0 .

$$\text{Thus} \quad \frac{x^{i+1}-1}{i+1} > \frac{x^i-1}{i} > \frac{x^{i-1}-1}{i-1} > \dots > \frac{x^j-1}{j}$$

if *j* is an integer $< i$.

$$\text{Hence} \quad (i-j)\{j(x^i-1) - i(x^j-1)\} \leq 0.$$

The extension to the cases in which *i* and *j* are replaced by any commensurable positive or negative fractions can then be carried out as in 'Chrystal.'

Remark.—The inequality has been demonstrated by expressing the excess of the greater over the less as the product of an essentially positive factor and the square of the difference between the two quantities (*x* and 1), which

being made equal will reduce the inequality to equality. Very many algebraic inequalities (perhaps all rational integral ones which become equalities for critical values of the quantities involved) can be demonstrated by expressing the difference in that form, or in a sum of expressions of that form. (See, for example, Roseveare's *Chapter*, § II.) R. F. MUIRHEAD.

REVIEWS.

A Mathematical Solution Book. By B. F. FINKEL. Pp. xvi., 549 (4th Edition.) (Kibler & Co., Springfield, Missouri, U.S.A.)

The care with which this little book has been written and its freedom from typographical errors have proved so acceptable to the youth of America that three editions of twelve hundred copies have now been exhausted,—and yet they want more. The author is the well-known Professor of Mathematics and Physics at Drury College, and his aim is to provide systematic solutions of difficult questions in the earlier subjects. He is very averse from “Short Cuts” and “Lightning Methods”; and insists that solutions should be written out step by step in logical order and the chain of reasoning made complete in every link. Possibly the logical instinct is more highly developed there than it is here, where the utter impossibility of comprehending the simplest reasoning is too often apparent in young pupils. “Do this, ye shall know why hereafter” is the only way [v. article in *Engineering* of June 19th, 1903, entitled “Reform in Mathematical Education,” inspired by Professor Perry].

Of 500 pages, 200 are devoted to Arithmetic, 200 to Mensuration (which evidently from its utilitarian nature looms very large indeed), 50 to Geometry (to which injury a further insult is added by filching 10 for Logic) and 50 to Algebra interspersed with Biographies.

The weights and measures used throughout the book are for the most part uncompromisingly English; but questions on money are worked in U.S. currency and on specific gravities in the metric system. Our old friend, the rod of $5\frac{1}{2}$ yards (whom we would so gladly spare in England) comes up smiling; and so does Apothecaries' weight. The most appalling accuracy prevails through the book,—the area of a certain circle is 12·566368 sq. ft. and π is given to 600 places of decimals (p. 262). How much more useful would be Professor Lodge's approximations for this and other multipliers! The largest known primes and perfect numbers are given on p. 42 (curious results, tricks to show the stretch of human brain), but one looks in vain for a table of prime numbers or factors. On pp. 554-5 there are tables of various constants, square and cube roots, natural logarithms, degrees into circular measure, etc.

We are perfectly willing to accept Mr. Workman's ironical statement that if we assume N to be the number required we are keeping within the limits of Arithmetic, whereas if we assume x to be the number we immediately remove our solution into the sphere of Algebra. What then must we think of this solution (p. 147)? “Had an article cost 10% less, the number of % gain would have been 15% more; what was the gain? Assume 100% = selling price, 100% = cost price. Then $1\frac{1}{2} \times \{100\% - (100\% - 10\%)\} = (100\% - 100\%) + 15\%!$ ”

Strictly speaking, there is an ambiguity about such an expression as “the percentage of marks gained was increased by 50%.” This may mean the percentage was increased (say) from 35% to 85%; or it may mean (say) from 40% to 60%. Perhaps in the latter case it would be more correct to say the percentage gained was increased by 50(%)!.

The addition, multiplication and division of fractions are carefully explained with illustrating scales. The writer evidently believes in the efficacy of a thorough drilling in these matters. “With a certain class of teachers, overcome with the desire to make everything in education easy, the complex fraction has fallen into disrepute. These teachers strongly advocate its omission . . . this is done, it is claimed, for the benefit of the student. As a matter of fact, it results in his eternal injury. . . It is pitiable to see average students in Analytical Geometry or Calculus struggling with Arithmetical operations—when they need all their energy to develop the principles of those subjects and should not be forced to dissipate any of it”—in the difficulties of calculation (p. 54).

We recognize some old favourites in Achilles and the Tortoise (p. 166) and the Hundred Cats Killing the Hundred Rats (p. 162). Some of the questions would seem strange to English readers: for instance (p. 416) "A gentleman proposed to plant a vineyard of 10A. If he places the vines 6 feet apart; how many more can he plant by setting them in the quincunx order than in the square order, allowing the plot to lie in the form of a square, and no vine to be set nearer its edge than 1 ft. in either case? Answer 1870."

From p. 233, where it is considered in what scale 1331 is denoted by 1000, we go right away in fifty pages to the rectification of the cycloid and quadrature of the spheroids—even triple and elliptic integrals being noted in the text. In many cases the integrations are given in the form of series available for small values of the variable. Facts and figures are given for the conchoid, limaçon, spirals, etc., and generally a great amount of useful work is crowded into the section of mensuration that in England we consider belongs to the Calculus. The length of the solutions is a little wearisome, and it would have been better to have used first approximations only in illustrating methods and principles.

The section on Geometry is very slight and inadequate. Believing (p. vi.) that "one solution, thoroughly analyzed and criticised by a class is worth more than a dozen seen through a cloud of obscurities," the Author gives one notable example of modern descriptive geometry:—If a rectangular hyperbola pass through the middle points of the sides of the triangle ABC and intersect those sides again in P, Q, R respectively, then AP, BQ, CR will co-intersect on the circum-circle. This is indeed an admirable and suggestive theorem. With its study we may well wrestle till break of day. Many interesting issues, such as one-to-one correspondence, are raised. Carnot's theorem teaches us that if a conic intersect the sides of ABC in $(P_1, P_2), (Q_1, Q_2), (R_1, R_2)$ and AP_1, BQ_1, CR_1 are collinear then so also are AP_2, BQ_2, CR_2 (a simple illustration is the Nine Point Circle). Such a conic has for its trilinear equation $\Sigma a/a_1 a_2 - \Sigma \beta \gamma / (\beta_1 \gamma_2 + \gamma_1 \beta_2) = 0$ (which is analogous to the Cartesian equation to the conic through four points). Make $aa_1 = b\beta_1 = c\gamma_1$, introduce the condition for a rectangular hyperbola, and we easily obtain $\Sigma a/a_2 = 0$.

At the end of the volume there are a dozen short biographies of living and dead worthies, those of Sylvester and Cayley being specially interesting. We recommend the Editor of the *Gazette* to consider whether he cannot, after some specially commendable article or solution, introduce a short up-to-date biography of the contributor.

On the whole the book is readable and instructive. The writer has keen enthusiasm for his science, "the practical applications of which have in all ages redounded to the highest happiness of the human race, and by its achievements has bound all the nations of the earth in one common brotherhood of man." Lest, however, any one should boast of being able to read far in nature's infinite book of secrecy let him lay to heart Newton's memorable words (quoted in the biography on p. 501), "I seem to have been only like a boy playing on the seashore and diverting myself in now and then finding a smoother pebble or prettier shell than ordinary; whilst the great ocean of truth lay all undiscovered before me."

R. F. DAVIS.

Lehrbuch der Theta-funktionen. By Dr. A. KRAZER. Teubner's "Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften," Band xii. Pp. xxiv. + 509. 24 m.

Dr. Krazer has succeeded in the difficult task of giving a clear deductive account of the complicated formal theory of multiple theta-functions within the compass of a moderate sized text-book. A sufficiently complete account is given of various allied and introductory theories to enable the reader to follow the main argument without much subsidiary study, while numerous references giving the historical development of the subject afford the means of elaborating particular details at pleasure.

The work well fulfils the requirements of a handbook and is distinguished by clearness of style and general elegance of form, though in the latter respect considerable improvement might be introduced by the use of matrix notation. The details of this notation are so easy to grasp that the common reluctance to employ it is remarkable. The singly infinite series $\Sigma \exp. (am^2 + 2mu)$ (the summation extending over all integral values of m) occurs for the first time in Fourier's

Théorie de la Chaleur, and its introduction as $\theta(u)$ into function-theory was one of the greatest advances made by Jacobi. The chief properties of this function are its uniformity, entirety, and certain periodic properties which, conversely, can serve to define it, and determine its usefulness in the construction of periodic functions. The general theta-function of p arguments may be defined by precisely the same expression where now α is a p -rowed square matrix and m, u are row-letters. To interpret, signs of summation and suffixes must be supplied, pairs of consecutive suffixes being the same; thus αm^2 , which should more correctly be written $m\alpha m$, means the sum of terms $m_i \alpha_{ij} m_j$. The great advantage of this notation lies not merely in its brevity but in the fact that the letters may be manipulated much like ordinary algebraical symbols, and the periodic properties proved for the single theta-function may be interpreted for the case of p arguments.

The first chapter develops this theory, introducing the "characteristic" and functions of higher order. The second chapter opens with the general theory of the linear transformation of the integers of summation in a multiply infinite series; the transformed series requires the number of solutions of a set of p linear congruences in a simple form involving the invariant-factors of the determinant of coefficients. Application is made of this to transform linearly the arguments and periods of the general theta-function and to obtain many general formulae.

After another modification of the theta-series by means of Fourier's expansion, follows one of the most interesting chapters in the book, showing how the general periodic function may be represented by the quotient of two theta-functions. The line of argument is based on the algebraic relation between $p+1$ functions having $2p$ systems of periods, and the relations among the periods deducible from this. The remaining two chapters of the first part of the book are devoted to Transformation and Complex Multiplication; the latter contains a convenient summary of the chief theorems concerning bilinear forms.

The second part deals mainly with half-integer characteristics (the coefficients in a set of half-periods added to the arguments) and the relations connecting different theta-functions. The particular cases of $p=1$ and $p=2$ are necessarily somewhat curtailed, but what is given is interesting and the list of references with brief descriptions is very comprehensive. In the case $p=2$ the various notations for the 16 theta-functions adopted by most writers are unfortunate in that they do not show clearly from what general formula particular relations are deduced, or, conversely, how to employ a formula without reference to a table. Dr. Krazer either writes the characteristic in full, which is inconvenient, or uses six different symbols for the odd characteristics and their combinations by threes for the even ones, thereby laying undue stress on a distinction which is not essential to the formal theory. If, however, a double suffix notation is used, which is suggested by the matrix notation for a half-period and divides the characteristic horizontally, instead of vertically as in Rosenhain's and Humbert's algorithms, it will be found that many formulae can be written in a general symbolic form from which particular instances can immediately be deduced.

The third part deals with Riemann's special theta-functions of one complex variable in which the p arguments have been replaced by the p finite Abelian integrals for an algebraic curve of deficiency p . The treatment of this portion suffers in respect to clearness and completeness by comparison with the great English treatise on the subject, but lack of space is responsible for many omissions.

It is scarcely necessary to mention that in a volume of this magnificent "Samm-lung" every care has been bestowed upon the details of printing so as to render the book as attractive as possible.

R. W. H. T. HUDSON.

Mechanics, Theoretical Applied and Experimental. By W. W. F. PULLEN, Wh. Sc., M.I.M.E., A.M.I.C.E. (Longmans, Green & Co.)

It is an unfortunate fact that most existing textbooks on elementary mechanics are written either, to borrow Professor Perry's classification, exclusively from the 'academic' or exclusively from the 'practical' standpoint; and are the poorer for this limitation. The best academic textbooks are admirably precise and clear, but unfortunately disregard entirely the historical and practical aspects of the subject. The baser kind of academic cram-book is equally silent as to matters of practical utility or historical interest, and is often disgracefully unsound on matters of fundamental principle, giving for instance a 'fudge' of Atwood's

machine by

$$\begin{aligned}\text{mass moved} &= m + m', \\ \text{moving force} &= (m - m')g.\end{aligned}$$

On the other hand, the practical textbooks, though they frequently draw attention to applications of great interest (and in this respect the book under review deserves high praise) are too often weak on the theoretical side, partly no doubt because they encounter a difficulty from which the purely 'academic' writer escapes.

A theory, embodied in a theorem, states that, given certain premises, a certain conclusion follows. This is all that the 'academic' writer is concerned with. For instance, if a weightless, flexible string passes round a smooth peg, the tensions on either side are the same. Now the constant repetition of these qualifying premises is tiresome, and there is a temptation to say "the tension of a string on either side of a pulley is the same." The original theorem is obviously inapplicable in practice; the modified form is free from this objection, but is untrue.

A writer, whose aim is practice, is tempted, adapting the medieval view that a doctrine may be theologically false, but philosophically true, to say that the proposition "the tension of a string is the same on either side of a pulley" is theoretically true, but practically false. The enunciation of conclusions which are based upon certain definite premises or conditions, as if they were absolutely and unconditionally true, is responsible for much of the contempt and dislike which the "practical man," to his own hindrance, is apt to show for theory.

Mr. Pullen states at page 51 and again (in block type) at page 68: "The mutual pressure between two surfaces in contact is perpendicular to the surfaces at the points of contact"; and at page 161: "The velocity of a body falling down any curved path is the same whatever the shape of the path."

The writer of a textbook should have the same aim as a conveyancer; each of his sentences should be capable of only one possible meaning. This should be so not merely for the sake of the subject, but for the sake of the English language. I should like to quote a few illustrative gems which are not from Mr. Pullen's book, but from other "practical works."

(i) "Velocity is said to be uniform when it moves over equal distances in the same time, and it is variable when it is continually altering."

(ii) "The radius of curvature in a beam of uniform strength and depth is a circle."

(iii) "Since generally force = mass $\times$ velocity (when force is measured by momentum)."

In Mr. Pullen's book, though it is generally carefully written, we come across a loose statement here and there. For example, the symbol g is used in three distinct senses without warning. On page 206 we find, "The student should try to remember that a force of g dynes is equivalent to a force of 1 gramme, and a force of g poundals is equivalent to a force of 1 pound." Could one severely blame a student who remembered that 32 dynes = 1 gramme? At page 157 and elsewhere we find "Force $\times g$ = mass moved $\times$ acceleration produced," a statement which, on the most favourable construction, is dangerous; for g , which heretofore denoted an acceleration, has suddenly become a mere number; and the transition is unexplained. There is some confusion between mass and weight at page 157, and on page 355 this leads to a positive mistake, for there we find, in block type, "Density = mass of unit of volume. Pressure of a fluid on a surface immersed in it = area of surface $\times$ depth of centre of gravity of surface $\times$ density of fluid."

In teaching the elements of mechanics to beginners, the maxim *Entia non sunt multiplicanda* should be translated: "Postpone the use of the word mass as long as you can."

Again, while fully admitting that a definition should generally be the coping stone rather than the foundation of a structure, I cannot consider the statement (p. 190), "The Radius of Gyration of a body is the distance from the axis of rotation at which the mass of a body can be supposed to be concentrated to produce the same result" as well fitted for either purpose. On page 42 we have a "general method for finding the resultant of any number of forces in one plane" without any warning of the difficulty which will arise if the system of forces happens to be equivalent to a couple. Such instances as these induce a feeling of lack of confidence; and even in exclusively practical matters Mr. Pullen is not always an absolutely reliable guide. On page 94 we read: "If the centre of gravity of

a balance is above the point of support . . . it would be altogether useless." But the so-called 'accelerating' balance is a very common object in shops. At page 23 the absence of an 'elastic' or 'plastic' stage in string is inferred from an experiment which appears rather to indicate the gradual pulling out of the 'twist.' It would be interesting to repeat the experiment with horsehair, unspun thread, and a stiff helical spring.

I have made the preceding criticisms because they relate to matters which can be put right by a little revision, and I now have the more pleasant task of pointing out the merits of the book. Much care has evidently been expended in selecting illustrations from real life of the mechanical principles discussed. To mention only three instances, that ingenious as well as useful appliance Weston's differential pulley is most thoroughly discussed with very numerous examples: the 'platform' weighing machine is well and clearly described, and directions are given for the repetition by Galileo's own method of his experiments on acceleration. Mr. Pullen lays much stress on actual experiment. There can hardly be two opinions as to the absolute justice of this view. Nothing is more disheartening than to find, as constantly happens, that boys have 'done mechanics,' and can accordingly sometimes work out a mechanical problem and sometimes obtain a correct numerical result, but give it a wrong description. When the reaction at a support is stated to be 400 foot pounds, or we are told * that "a cubic foot of water moving at 113 feet per second has momentum equal to 798,062 foot lbs.," we have a demonstration that in the pupil's view, force, momentum, energy, acceleration, and so on, are only words, and obviously, in the republic of letters, one word is as good as another.

Though, for the reasons I have stated, the book is not one to put unreservedly into the hands of a beginner until it has undergone revision; yet no teacher can take it up without deriving much benefit from the numerous valuable hints and examples which it contains. The book is remarkably well printed with clear and excellent diagrams.

C. S. JACKSON.

Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung Statistik und Lebensversicherung. Vol. II. By EMANUEL CZUBER. 1903. (Teubner.)

"The calculation of probabilities and their application to the adjustment of error, Statistics and Life Insurance," is a well-printed book, setting forth clearly and concisely the principles underlying the science of Life Insurance. Commencing with an exposition of the mathematical probabilities of death and survivorship, the author goes on to explain the functions known as the force of mortality and the central death rate, and the method of calculating the expectation of life. Then follow some interesting methods of representing geometrically the effects of mortality upon masses of individuals, methods differing from those employed by English actuaries.

The systems of constructing and graduating mortality tables are set forth at length. The graduation formulas of Gompertz, Makeham, Wittstein, Woolhouse, and Karup are analysed, and the graphic method advocated by Dr. Sprague is also explained.

The chapter on Invalidity and Mortality is interesting, as it deals with a subject of special importance in Germany, and not unlikely in future to call for consideration in this country. The analysis of the formulas for calculating annuity values is clear and complete, but we think there is a slight typographical error on page 458, where $m\left(\frac{1}{v^m} - 1\right)^{\frac{1}{m}}$ is put equal to $m\{(1+i)^{\frac{1}{m}} - 1\}$. Formulas

are also given for annuities payable during "Activity" and "Invalidity." It is noteworthy in this connection that the commutation column N_x is based on the relation $N_x = D_x + D_{x+1} + D_{x+2} + \text{etc.}$ instead of $N_x = D_{x+1} + D_{x+2} + \text{etc.}$, as used in this country, and this also in the case of the Hm. Mortality Table of the 20 British offices in which the text-book graduation is adopted. Seeing that the column of values of μx from this table is also given, we are somewhat surprised that no mention is made of the very valuable formulas of approximate

* This statement got into print, but is a good example of the sort of thing I refer to.

summation applied to actuarial purposes by Mr. G. F. Hardy and introduced by Mr. King into Part II. of the text-book of the Institute of Actuaries.

Zillmer's method of dealing with the problem of providing for the special expenditure incurred in the first year of insurance—a problem to which Dr. Sprague among English actuaries has devoted especial attention—is given, the basis being as follows:—"The insured pays annually an equal gross premium whereas the net premium is constant from the second year, but calculated as less for the first year than the following years by an amount a . This amount which enters into the additional payment, and may therefore serve for other than pure insurance purposes, is intended to cover the preliminary expenses. The net premium for the first year, however, must, if the first principles of variable premium payments be satisfied, be at least equal to the natural premium. Obviously the reduction of the first premium involves a corresponding increase of the following premiums. The notion underlying this method of calculation may be amplified in that the first addition may be spread over several years instead of over one. Dealing, for example, with a whole life insurance with a premium payable annually throughout life; from the second year the premium amounts to P_x , then in the first year it is $P_x - a$, and the present value of all the premiums is expressed by $P_x a_x - a$. Equating this to the value A_x of the insurance we have

$$P_x = \frac{A_x + a}{a_x}, \text{ while the premium remaining constant throughout the whole term is } P'_x = \frac{A_x}{a_x}.$$

The first essential requirement is that a must be taken less than P_x . This leads to the condition $a < \frac{A_x + a}{a_x}$.

$$\begin{aligned} a \left(1 - \frac{1}{a_x} \right) &< P'_x, \\ a &< \frac{a_x}{a_x - 1} P'_x \dots\dots\dots (1) \end{aligned}$$

The second requirement formulated above is $P_x - a \geq \frac{C_x}{D_x}$.

$$\begin{aligned} P'_x + \frac{a}{a_x} - a &\geq \frac{C_x}{D_x}, \\ a &\leq \frac{a_x}{a_x - 1} \left(P'_x - \frac{C_x}{D_x} \right) \dots\dots\dots (2) \end{aligned}$$

If (1) is fulfilled (2) is fulfilled also; hence the right hand member of (2) becomes an upper limit for the amount a , a limit which varies (increasingly) with the age. It is another question whether a selected with this limit is always sufficient to cover the preliminary expenses or whether this is at least to be obtained with the aggregate of the insurances."

It must be borne in mind that in the notation employed throughout the book $a_x = \frac{D_x + D_{x+1} + \text{etc.}}{D_x}$, or $1 + a_x$ in the English notation, it is the value of an annuity due; hence in the English notation formula (2) becomes

$$a \leq \frac{1 + a_x}{a_x} \left(P'_x - \frac{C_x}{D_x} \right).$$

From the Hm. Table the upper limit of a when $x=30$ is given as $a \leq 0.01033$, about 1% of the capital sum assured. When $x=50$, $a \leq 0.02321$, about 2.3% of the capital sum assured.

Next follow chapters on insurances with guaranteed return of premiums paid and on policy values, the latter containing a brief explanation of the formulas applicable to a change in the form of an insurance; for example, from a whole life to an endowment assurance. The final chapter deals with the mean risk in life insurance and the solution of the general problem of estimating the risk incurred by an insurance office as regards its insured at a given point of time. The problem is dealt with at some length, and is likely to be of interest to English actuaries.

Appended to the book are the following tables :—The value of the function $\phi(\gamma) = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2} dt$; the German mortality table for male and female lives; the

Hm. Table 20 offices $3\frac{1}{2}\%$; Tables M. and W. I. of 23 German offices $3\frac{1}{2}\%$; and invalidity and mortality tables of German railway officials $3\frac{1}{2}\%$.

The volume is deserving of careful study, and is ably compiled. It is not entirely free from printer's errors, but these are commendably few, having regard to the difficult nature of the subject and the necessarily complicated nature of the symbols employed.

W. R. STRONG, F.I.A.,
London Guarantee and Accident Co.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor*. 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.).]

The Mathematical Monthly. Vols. I-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. Complete. Vols. 1-35. Half leather. £25.

Also Vols. 25-35 of the same. Cloth. Half price, £7.

(2) Wanted.

Vols. I-IV. *Mathesis*. 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

Tortolini's Annali. Vol. I. (1850), or any one of the first eight parts of the Volume.

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Tōkyō Sugaku-Buturigaku Kwai Kizi. Maki. No. IX. Dai 1.

On the Isosceles Trapezium. T. Hayashi. *The Formula and Tables for finding the time with a portable transit instrument in the vertical circle of Polaris (or λ Ursae Minoris)*. H. Kimura.

Notes on Two Intrinsically Related Plane Curves. By R. F. DAVIS. pp. 4 (from *Proc. Edin. Math. Soc.*, Vol. XXI.). 1903.

Divergence and Divergive Power in Elementary Optics. By R. F. MUIRHEAD. pp. 9. (*Proc. Roy. Phil. Soc.*, Glasgow.) 1903.

De latine sine flexione. Principio de permanentia. By G. PEANO. pp. 14. (*Rev. di Math.* 1903. tom. 8.)

Zeitschrift für Geschichte der Mathematischen Wissenschaften. Sonderabdruck aus Bibliotheca Mathematica. III. Folge IV. 1903. *Peter Guthrie Tait; His Life and Works*. By A. MACFARLANE. pp. 185-200. V.

Elementary Geometry, Practical and Theoretical. By C. GODFREY and A. W. SIDDONS. pp. xi., 355. 1903. 3s. 6d. (Cambridge University Press.)

A New Geometry for Schools. By S. BARNARD and J. M. CHILD. pp. xxvi., 512. 1903. 4s. 6d.

Elementary Algebra. Part I. By C. MUKERJEE. pp. 205, 34. 1903. (The Indian Press, Allahabad.)

Vectors and Rotors with Applications. By O. HENRICI and G. C. TURNER. pp. xv., 204. 4s. 6d. 1903. (Edward Arnold.)

A Short Course of Arithmetic. By A. E. LAYNG. viii., 220. 1s. 6d. 1903. (Blackie.)

BOOKS, ETC., RECEIVED.—Continued.

- Essai sur l'Hyperspace.* By M. BOUCHER. pp. 204. 2 fr. 50 c. 1903. (Alcan.)
- A School Geometry.* Part III. (Circles). By H. S. HALL and F. H. STEVENS. With Answers. pp. viii., 137-210, vi. 1s. 1903. (Macmillan.)
- The Journal of the College of Science, Imperial University of Tōkyō, Japan.* Vol. XIX. Art. 5.
- Ueber die im Bereiche der rationalen complexen Zahlen Abel'schen Zahlkörper.* By Prof. T. TAKAGI.
- Sur la Philosophie des Mathématiques.* By J. RICHARD. pp. 248. 2 fr. 50 c. 1903. (Gauthier-Villars.)
- Analyse Infinitésimale à l'usage des Ingénieurs.* By E. ROUCHÉ and L. LÉVY. Vol. II. pp. x., 848. 15 frs. 1903. (Gauthier-Villars.)
- The Annals of Mathematics.* Edited by ORMOND STONE and others. Second Series. Vol. IV. No. 4. 2s. (Longmans.)
- On the Characteristics of Differential Equations.* E. R. HEDRICK. *Convergence of Series.* M. BÖCHER. *Integral as Limit of Sum.* W. F. OSGOOD. *A General Relation of Continued Fractions.* R. E. MORITA. *On the Equilateral Hyperbola.* J. A. VAN GROOS. *The Generalised Wilson's Theorem.* G. A. MILLER. *A Sufficient Condition for the Maximum Number of Roots of an Equation of the n^{th} degree.* E. B. VAN VLECK.
- American Journal of Mathematics.* Edited by FRANK MORLEY. Vol. XXV. No. 3. 1½\$. July 1903. (Kegan Paul.)
- Isothermal Conjugate Systems of Lines on Surfaces.* L. P. EISENHART. *Some Differential Equations connected with Hypersurfaces.* G. O. JAMES. *Forms of Sextic Scrolls of Genus greater than One.* V. SNYDER. *Geometry on the Cuspidal Cubic Cone.* F. C. FERRY.
- Elemente der Vector-Analysis.* By A. H. BUCHERER. pp. vi., 91. 1903. (Teubner.)
- Mathematischer Bücherschatz.* By E. WÖLFFING. Part I. *Reine Mathematik.* pp. xxxvi., 416. 1903. (Teubner.)
- Einleitung in die Allgemeine Theorie der Algebraischen Grössen.* By J. KÖNIG. pp. x., 564. 1903. 18 m. (Teubner.)
- Vorlesungen über Algebra.* By G. BAUER. pp. vi., 376. 1903. 13 m. (Teubner.)
- Rechenbuch für die unteren Klassen der Höheren Lehranstalten.* By H. MÜLLER and F. PIEKKER. pp. viii., 244. 1903. 2 m. 40. (Teubner.)
- Methodisches Lehrbuch der Elementar-Mathematik.* By G. HOLZMÜLLER. pp. xiv., 370. 1903. 4 m. 40. (Teubner.)
- Arithmetic.* By J. ALISON and J. B. CLARK. Part I. pp. viii., 304, xxxvi. 1903. 2s. 6d. (Oliver & Boyd.)
- A Course of Pure Geometry.* By E. H. ASKWITH. pp. viii., 208. 1903. (Cam. Univ. Press.)
- The Monist.* July, 1903. 2s. 6d. (Open Court, Kegan Paul.)
- The Foundations of Geometry (concluded).* By PAUL CARUS.
- Théorie Nouvelle des Fonctions, exclusivement fondée sur l'Idée de Nombre.* By G. ROBIN. pp. vi., 216. 7 frs. 1903. (Gauthier-Villars.)
- First Stage Practical, Plane, and Solid Geometry.* By G. F. BURN. pp. viii., 240. 2s. 1903. (Univ. Tutorial Press.)
- Theoretical Geometry for Beginners.* By C. H. ALLCOCK. Part II. pp. 124. 1s. 6d. 1903. (Macmillan.)
- The Arithmetic of Physics and Chemistry.* By H. M. TIMPANY. pp. 74. 1s. 1903. (Blackie.)
- Wiadomości Matematyczne.* Tom. VII. Zeszyt 3-4. Edited by S. DICKSTEIN. 1903. (Warsaw.)
- Sullo Stato della Teoria Congruenze Binomie avanti il 1852.* By ALASIA CRISTOFORO. pp. 32. *Riv. di Fis. Mat. e Sc. Nat.* No. 44. 1903. (Pavia.)
- Algunas Observaciones sobre Fórmulas de las Superficies.* By ALASIA CRISTOFORO. *Revista Trimestral de Matemáticas.* Nos. 9-11. 1903. (Zaragoza.)
- Proceedings of the Edinburgh Mathematical Society.* Vol. XXI. 1902-1903.
- Mathematical Correspondence (R. Simson, Matthew Stewart, James Stirling).* Dr. MACKAY. *Proofs of Fourier's Series.* Dr. CARLAW. *Generalised Forms of the Series of BESSEL and LEGENDRE.* Rev. F. H. JACKSON. *On Uniqueness of Solution of the Linear Differential Equation of the Second Order.* Dr. FIDDLE. *On Two Intrinsically Connected Plane Curves.* R. F. DAVIS. *On Anti-reciprocal Points.* A. G. BURGESS. *On Singular Points of Plane Curves.* T. B. SPRAGUE. *Triangles in Multiple Perspective.* J. A. THIRD. *Mechanical Construction of Quartic Trisectrix.* H. POOLE. *Identities and Inequalities of Symmetric Algebraic Functions of n Letters.* R. F. MUIRHEAD, ETC.

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament thereunto."--BACON.

President :

Professor A. R. FORSYTH, LL.D., Sc.D., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.	R. LEVETT, M.A.
Prof. W. H. H. HUDSON, M.A.	Professor A. LODGE, M.A.
Professor G. M. MINCHIN, F.R.S.	

Treasurer :

F. W. HILL, M.A., City of London School, London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.
H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

S. BARNARD, M.A.	F. S. MACAULAY, M.A., D.Sc.
R. F. DAVIS, M.A.	W. N. ROSEVEARE, M.A.
J. M. DYER, M.A.	A. W. SIDDS, M.A.
C. GODFREY, M.A.	C. O. TUCKER, M.A.
G. HEPPEL, M.A.	A. E. WESTERN, M.A.
C. E. WILLIAMS, M.A.	

THE MATHEMATICAL ASSOCIATION, formerly known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all masters and mistresses, who are interested in promoting good methods of mathematical teaching. Such an Association should be a recognized authority in its own department, and should exert an important influence on methods of examination.

General Meetings of the Association are held in London once a term, and in other places if desired. At these Meetings papers on elementary mathematics are read, and any member is at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

"The *Mathematical Gazette*" is the organ of the Association. It contains—

- (1) ARTICLES, on subjects within the scope of elementary mathematics.
- (2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books.
- (3) REVIEWS, at present the most striking feature of the *Gazette*, and written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell rather on the general development of the subject, than upon the part played therein by the book under notice.
- (4) PROBLEMS AND SOLUTIONS, generally selected to show the trend of investigation at the universities, so far as is shown in the most recent scholarship papers. Questions of special interest or novelty also find a place in this section.
- (5) SHORT NOTICES, of books not specially dealt with in the REVIEWS.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "The *Mathematical Gazette*."

JAN 18 1904

THE

MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.

E. T. WHITTAKER, M.A.

W. E. HARTLEY, B.A.

LONDON

GEORGE BELL & SONS, YORK ST., COVENT GARDEN
AND BOMBAY

Vol. III., No. 43.

JANUARY, 1904.

1s. 6d. Net.

The Mathematical Association.

THE ANNUAL MEETING of the MATHEMATICAL ASSOCIATION will be held at *Two p.m. on Saturday, 23rd January, 1904*, at King's College, Strand, W.C.

PAPERS.

- (1) An account of a recent discussion on the possibility of a fusion of the Teachers of Mathematics and Science. Mr. C. S. JACKSON.
- (2) A Geometrical Note. Mr. J. C. PALMER.
- (3) Models of Regular and Semi-regular Solids, including the four polyèdres étoilés of Poinot. Mr. E. M. LANGLEY.
- (4) Advanced School Courses of Mathematics. Mr. C. A. RUMSEY.

The Mathematical Gazette is issued in January, March, May, July, October, and December.

Problems and Solutions should be sent to

W. E. HARTLEY, 6 Benson Street, Cambridge.

All other correspondence concerning the Gazette should be addressed to

W. J. GREENSTREET, Marling Endowed School, Stroud, Glos.

Correspondence relative to the Mathematical Association should be addressed to one of the Hon. Secretaries:—C. PENDLEBURY, St. Paul's School, London, W.; H. D. ELLIS, 12 Gloucester Terrace, Hyde Park, W.

If Copies of the Gazette fail to reach a member in consequence of a change of address, of which the Secretaries have not been informed, duplicate copies cannot be supplied, except on payment of 1s. per copy.

CONTENTS.

	PAGE
ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE. Prof. A. LODGE, ...	373
DRAFT SUGGESTIONS OF THE SUB-COMMITTEE ON THE TEACHING OF MECHANICS, ...	390
MATHEMATICAL NOTES. R. F. DAVIS; W. O. HEMMING, M.A.; H. HILTON, M.A., ...	383
PROBLEMS, ...	384
SOLUTIONS, ...	385
REVIEWS, ...	387
CORRESPONDENCE, ...	396
NOTICE, ...	396
QUERIES, SALE, AND EXCHANGE, ...	396
BOOKS, ETC., RECEIVED, ...	<i>Inside back cover</i>

Forthcoming Articles and Reviews by:—

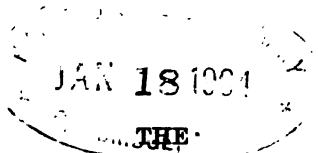
Prof. T. J. I'a BROMWICH; Prof. A. LODGE; Prof. G. B. MATHEWS, F.R.S.; J. H. GRACE, M.A.; R. W. H. T. HUDSON, M.A.; E. M. LANGLEY, M.A.; W. P. WORKMAN, M.A.; H. W. RICHMOND, M.A.; A. N. WHITEHEAD, M.A.; E. W. BARNES, M.A.; F. S. MACAULAY, D.Sc., and others.

BACK NUMBERS OF THE MATHEMATICAL GAZETTE.

THE change from quarto form was made with No. 7. No. 8 is out of print. A few numbers may be obtained from the Editor at the following prices, post free: No. 7, 5s.; Nos. 9-18, 1s. each.

The set of 7, 9-18, 15s.

These are now out of print and very scarce.



MATHEMATICAL GAZETTE.

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;

E. T. WHITTAKER, M.A.; W. E. HARTLEY, B.A.

LONDON :

GEORGE BELL & SONS, YORK ST., COVENT GARDEN,
AND BOMBAY.

VOL. III.

JANUARY, 1904.

No. 43.

ON THE REPRESENTATION OF IMAGINARY POINTS BY REAL POINTS ON A PLANE.

(Continued from the Mathematical Gazette for May, 1903.)

I wish to express my great indebtedness to Mr. P. J. Heawood, of Durham University, for his careful and suggestive criticism of the contents of this paper, without, however, making him responsible for any statement in it with which he may disagree.—
A. LODGE.

For the sake of those readers who have not seen the first part of this paper, and to whom the whole subject is new, I may state that the whole argument is based on the fact that any imaginary point such as $(6+i, 4+3i)$ has a simple vector $x+iy = (6+i) + i(4+3i) = 3+5i$, and therefore may be vectorially plotted on to, or represented by, the real point [in this case $(3, 5)$] which has the same vector. This real point is thus the *representative* of all the imaginary points which have the same vector. The paper deals with the consequent relations between the representatives of conjugate imaginaries, especially when the imaginaries satisfy the equation of some given curve: in this case the two real points representing such a pair of conjugate imaginaries are called images of each other in the curve.

Every point P in the plane has n such images in a curve of the n^{th} degree. If two images of P coincide, P is a focus—unless the two images coincide with P itself, in which case it is a double point on the curve.

THERE are a number of interesting theorems relating to the pair of representatives of a pair of conjugate imaginary points.

The first thing to do is to establish certain fundamental equations by means of which when any two of the four points are given the others can be found.

Let P, Q be the real points, with coordinates (α, β) and (γ, δ) respectively; and let I, J be the corresponding imaginary points with coordinates (x_1, y_1) and (x_2, y_2) respectively.

Then, since by supposition an imaginary point and its representative have the same *vectors*, we have the two equations

$$x_1 + iy_1 = \alpha + i\beta, \dots\dots\dots(1)$$

$$x_2 + iy_2 = \gamma + i\delta. \dots\dots\dots(2)$$

and

T

Two other important equations can be obtained by considering the values of what may be called the anti-vectors of the points, i.e. the values of $x - iy$.

It is obvious that $x_1 + iy_1$ and $x_2 - iy_2$ differ throughout in the sign of i , and in that only; for if the coordinates of I are of the form $m + ip$, $n + iq$, those of J will be of the form $m - ip$, $n - iq$.

Hence, $x_2 - iy_2 = \alpha - i\beta$, ...(3)

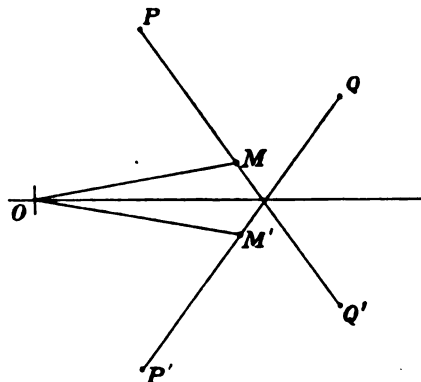
and, similarly, $x_1 - iy_1 = \gamma - i\delta$(4)

These four equations are sufficient to find the other two points when two of them are given.

They can be illustrated graphically as shown below; for though the imaginary points themselves cannot be shown (unless they are considered as coinciding with the representative points), their *coordinates* can be so represented, since x and iy are merely vectors whose addition leads us to the point (x, y) .

Let

P be the point (α, β) ,
 P' " " $(\alpha, -\beta)$,
 Q " " (γ, δ) ,
 Q' " " $(\gamma, -\delta)$.



Then, if M is the mid-point of PQ , we have, by addition of vectors,

$$OM + MP = OP,$$

$$OM - MP = OQ'.$$

These are the equations:

$$x_1 + iy_1 = \alpha + i\beta, \text{(1)}$$

$$x_1 - iy_1 = \gamma - i\delta. \text{(4)}$$

Hence

$$x_1 \text{ is the vector } OM,$$

and

$$iy_1 \text{ " " } MP.$$

Similarly, if M' is the mid-point of $Q'P'$, we see, from equations (2) and (3),

that

$$x_2 \text{ is the vector } OM',$$

and

$$iy_2 \text{ " " } M'Q.$$

[In the diagram, P is at $(3, 5)$, and Q is at $(9, 3)$; and the coordinates of I are $(6 + i, 4 + 3i)$; while those of J are $(6 - i, 4 - 3i)$. So that the vector of

$$OI = 6 + i + i(4 + 3i) = 3 + 5i = OP,$$

and the vector of $OJ = 6 - i + i(4 - 3i) = 9 + 3i = OQ.]$

It follows, from the above, that any pair of points P, Q in the plane can represent some one pair of conjugate imaginary points, and also that to any point P correspond an infinite number of imaginary points given by the equation (1), the conjugates of such points having representative points Q , which cover the whole plane.

Further, there will be one imaginary point given by equation (1) whose coordinates will be infinite, and the conjugate of that point will be situated at infinity. These points, which may be called Ω and Ω' , are the circular points at infinity. All the points given by equation (1), including Ω , appear to be piled on P , and Ω itself having infinite coordinates may be considered as being the meeting point of all the lines $x + iy = \text{constant}$.

If any pair of conjugate imaginary points I, J are points of a curve $f(x, y) = 0$, their representative points, P, Q , will, as a rule, not be situated on the curve, but they will have in many cases interesting relations with the curve. I propose to call them images of each other in the curve. In a curve of the n^{th} degree we can show that there will be n imaginary points of the curve, $I_1, I_2, \dots, I_n$, corresponding to any given representative point P which is not itself a point on the curve, and these points will have n conjugate points $J_1, \dots, J_n$, whose representative points $Q_1, \dots, Q_n$ will be scattered over the plane, i.e. every point P will have n images in a curve of the n^{th} degree.

For, if $\alpha + i\beta$ be the vector common to any given point P and to all the I points represented by P , then if we denote the vector of one of the conjugate J points by z and its anti-vector by ω , we shall have, for any of these J points

$$\begin{aligned}x + iy &= z, \\x - iy &= \alpha - i\beta = \omega;\end{aligned}$$

ω being given, and z being required.

From these we obtain

$$\begin{aligned}2x &= z + \omega, \\2iy &= z - \omega; \\\therefore 2y &= i(\omega - z).\end{aligned}$$

Substituting these values in $f(x, y) = 0$, we obtain the required equation in z , viz.:

$$f\left\{\frac{\omega + z}{2}, \frac{i(\omega - z)}{2}\right\} = 0,$$

the solution of which gives the vectors of the images of P , since J and Q have the same vector.

In this equation, the highest power of z involved is z^n , consequently the equation will be of the n^{th} degree unless the highest terms cancel. In such exceptional case, one or more of the values of z must be infinite, and consequently the points Ω and Ω' will be imaginary points of the curve.

Including the infinitely distant points, we may therefore say, generally, that to each point P correspond n images $Q_1, \dots, Q_n$. These points represent n imaginary points $J_1, \dots, J_n$ of the curve, the conjugates of which are $I_1, \dots, I_n$, all of which have the same vector as P .

[The above equation between z and ω may be called the z, ω equation of the curve, z and ω being respectively the vector and anti-vector of any point of the curve. It is important to notice, however, in the case of an imaginary point, that whereas z is also the vector of the real representative of the imaginary point, ω is the anti-vector of the real representative of the conjugate imaginary point.]

Various interesting special cases suggest themselves. Thus, if P is on the curve, one of the images will coincide with it; if it is a double point on the curve, two of its images will coincide with it, and so on. It is possible, however, for two of the images to coincide with each other without

coinciding with P , whether P is on the curve or not; in such cases P will be a focus of the curve, for the definition of a focus is that two of its I points are coincident,* which necessitates the corresponding Q points being also coincident. If one or more images of every point be at infinity, the point, if any, which has *another* infinitely distant image will be a focus.†

Again, considering special curves, I showed in my first paper that if $f(x, y) = 0$ is a straight line, any pair of images, P and Q , are perpendicularly equidistant from the line on opposite sides of it; i.e. they are, as it were, 'optical' images in the line, the distance of each from the line being equal to the modulus of the purely imaginary part of the coordinates of the corresponding imaginary points I and J . This line may be called the axis of the pair of conjugate points I and J . Hence every imaginary point I has a representative point P and an axis, the axis being the perpendicular line midway between P and its image Q (Q being the representative of the conjugate imaginary point J).

If $f(x, y) = 0$ is a circle, we can show, as follows, that P and Q are inverse points with regard to the circle. For, taking the equation $x^2 + y^2 = r^2$, we find that its z, ω equation is $z\omega = r^2$.

Hence, if P is at (α, β) , and ω is its anti-vector $\alpha - i\beta$, z will be the vector of Q , and is given by the equation

$$z = \frac{r^2}{\omega} = \frac{r^2}{\alpha - i\beta} = \frac{r^2}{\alpha^2 + \beta^2} (\alpha + i\beta).$$

Therefore Q is at $\left(\frac{r^2\alpha}{\alpha^2 + \beta^2}, \frac{r^2\beta}{\alpha^2 + \beta^2} \right)$, which is obviously the inverse of (α, β) .

It is important to notice that although a circle is a curve of the second degree, each point in the plane has only one finite image, the equation in z being of the first degree. The other image is at an infinite distance. The reason is that though x^2 and y^2 both contain z^2 , $x^2 + y^2$ does not, being equal to $z\omega$, the z^2 terms cancelling out. Thus Ω and Ω' are imaginary points of the circle.

In the same way, any curve for which every point in the plane has an infinitely distant image must have $x^2 + y^2$ as a factor of the terms of the n^{th} degree, where n is the degree of the curve, and every point will have two infinitely distant images if $(x^2 + y^2)^2$ is a factor of the n^{th} degree terms, and if either the terms of the $(n-1)^{\text{th}}$ degree are missing, or $x^2 + y^2$ is a factor of them. The first set of curves are called circular, since Ω and Ω' are imaginary points of them; and the second set are called bicircular, Ω and Ω' being imaginary double points of them. Similarly, we may have tricircular curves, and so on. In any of these cases, if any special point has *another* infinitely distant image, that point will be a focus, as already stated.

The conditions for foci may be illustrated by finding the vector equation of the images of any point (α, β) in the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, and showing that they will be coincident if (α, β) is a focus, and in no other case.

The z, ω equation of the ellipse is

$$b^2(z + \omega)^2 - a^2(z - \omega)^2 = 4a^2b^2,$$

which reduces to

$$z^2 - 2 \frac{a^2 + b^2}{a^2 - b^2} z\omega + \omega^2 + \frac{4a^2b^2}{a^2 - b^2} = 0,$$

whence

$$z = \frac{a^2 + b^2}{a^2 - b^2} \omega \pm \frac{2ab}{a^2 - b^2} \sqrt{(\omega^2 - a^2e^2)}.$$

* In other words, (α, β) is a focus if $x + iy = \alpha + i\beta$ is a tangent, which necessitates $x - iy = \alpha - i\beta$ being also a tangent.

† In this case $x + iy = \alpha + i\beta$ is an asymptote in the direction of Ω , and $x - iy = \alpha - i\beta$ is an asymptote in the direction of Ω' .

This equation gives the required images, and shows that they will be coincident if $\omega = \pm ae$; i.e. if $a = \pm ae$, $\beta = 0$, and in no other case: which proves the proposition.

In finding the points of intersection of two curves, if any of the points are imaginary they will occur in pairs of conjugate points, whose real representatives will be images in each of the curves, as above defined. If we work with vectors, these real points will be given by the same equation as if they were points on the curves.

The method will be to eliminate x and y between the equation $z = x + iy$ and the equations of the two curves. The resulting equation in z will give the vectors of all the points of intersection, real and imaginary.

We will take as an important illustration of this method the following problem:

To find the pair of images common to an ellipse and one of its directrices, and to show that the director circle of the ellipse has the same two points as images.

Taking the three equations $z = x + iy$,

$$b^2x^2 + a^2y^2 = a^2b^2,$$

$$ex = a,$$

we easily find

$$z = \frac{a}{e} + \frac{b^2}{ae} = ae, \text{ or } \frac{a^2 + b^2}{ae};$$

$\therefore$ the required pair of images are $(ae, 0)$ and $\left(\frac{a^2 + b^2}{ae}, 0\right)$.

It is obvious that the equation $x^2 + y^2 = a^2 + b^2$; i.e. $zw = a^2 + b^2$, is satisfied by the vector and anti-vector of each of the imaginary points represented by these same points.

This problem has been introduced because it connects the foci of the ellipse with its director circle.

We have seen that the foci of a curve are points which have two coincident images in the curve. They are distinguishable from double points on the curve by the fact that the double images are not coincident with the points themselves. We will now find the point or points which have two coincident images in any curve $f(x, y) = 0$.

In this case it will be more convenient to denote the vector of the point itself by z , and to use ω to denote the anti-vector of its image.

The z, ω equation of the curve is

$$f\left\{\frac{\omega + z}{2}, \frac{i(\omega - z)}{2}\right\} = 0.$$

Denoting this, for brevity, as $u = 0$, we must have, as the condition for two equal values of ω , the two simultaneous equations,

$$u = 0, \quad \frac{du}{d\omega} = 0.$$

Now $\frac{du}{d\omega} = \frac{du}{dx} \cdot \frac{dx}{d\omega} + \frac{du}{dy} \cdot \frac{dy}{d\omega}$, z being constant,

where

$$2x = z + \omega$$

and

$$2iy = z - \omega;$$

$$\therefore \frac{du}{d\omega} = \frac{1}{2} \left(\frac{du}{dx} + i \frac{du}{dy} \right).$$

T 2

Hence we may write the two simultaneous equations as

$$\left. \begin{aligned} u &= 0, \\ \frac{du}{dx} + i \frac{du}{dy} &= 0. \end{aligned} \right\}$$

We may, if we please, solve these as they stand and then plot the required points by the values of the vector $z = x + iy$, or we may use their z , ω equations, and eliminate ω between them. The resulting equation in z will, in either case, give the required points, which will include the foci and any double points the curve may have.

N.B.—Since $\frac{du}{dx} + i \frac{du}{dy} = 0$ is satisfied by the coordinates of the imaginary points on the curve corresponding to the foci, it follows that $\frac{du}{dx} - i \frac{du}{dy} = 0$ is satisfied by the coordinates of the conjugate imaginary points. Hence, both the foci and their images are given by

$$\left. \begin{aligned} u &= 0, \\ \left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 &= 0, \\ z &= x + iy, \end{aligned} \right\}$$

and any curve of the form

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = A_1 u + A_2 u^2 + \dots$$

will have the same points as images.

We will apply this method to find the director circle and directrices, and also the foci and focal images of the conic

$$2x^2 - 2xy + 2y^2 - 2x - 8y + 11 = 0; \dots\dots\dots(1)$$

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 0 \text{ gives}$$

$$(2x - y - 1)^2 + (x - 2y + 4)^2 = 0. \dots\dots\dots(2)$$

If we eliminate the xy term between (1) and (2) we obtain for the director circle

$$x^2 + y^2 - 4x - 6y + 9 = 0. \dots\dots\dots(3)$$

If from (1) and (2) or from (1) and (3) we make the second degree terms a perfect square, which can be done in two ways, we obtain the directrices $(x+y-8)(x+y-2)=0$, and a pair of imaginary directrices (parallel to the major axis) $x-y+1=\pm i$.

Solving these simultaneously, we obtain four points whose vectors are

$$z=i, \quad z=1+2i, \quad z=3+4i, \quad z=4+5i.$$

Hence the two inner points, viz. (1, 2) and (3, 4), are the foci; and their images are (0, 1) and (4, 5).

The equation of the major axis is $x-y+1=0$, and that of the minor axis is $x+y-5=0$, these being the lines midway between the two imaginary and the two real directrices respectively.

Applying the same method to the general conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

we obtain for the director circle

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4(a+b)u,$$

which reduces to the normal form

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0.$$

The real and imaginary directrices are given by

$$\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 = 4\lambda u,$$

where λ has one or other of the values given by the quadratic equation

$$(a - \lambda)(b - \lambda) = h^2.$$

If we wish to obtain the foci *without* their images, we eliminate x, y between

$$u = 0,$$

$$\frac{du}{dx} + i \frac{du}{dy} = 0,$$

$$x + iy = z,$$

and find for the vector equation of the foci

$$Cz^2 - 2(G + iF)z + (A + 2iH - B) = 0.$$

[NOTE.—In the above, the capital letters $A, B, \dots$ denote the first minors of $a, b, \dots$ in the discriminant of the conic.]

In the foregoing work we have tacitly assumed that the position of the point P which represents an imaginary point I of any curve is fixed relatively to the curve, i.e. is independent of any particular system of rectangular axes. It is perhaps desirable to prove this formally. This we shall do by showing that the position of the representative point is not affected by (1) changing the origin, (2) rotating the axes.

(1) *Move the origin to (h, k) .*

Let (α, β) be the original coordinates of P , and (x, y) the original coordinates of the corresponding imaginary point I , so that $x + iy = \alpha + i\beta$.

Then the new coordinates are respectively $(\alpha - h, \beta - k)$ and $(x - h, y - k)$. We have to show that these continue to possess equal vectors.

Now, the new vector of I is $(x - h) + i(y - k)$ which

$$= (\alpha - h) + i(\beta - k) \text{ since } x + iy = \alpha + i\beta.$$

Therefore the new vector of P is equal to the new vector of I .

Hence P is still the representative of I .

(2) *Rotate the axes through the angle ϕ .*

Let $(r \cos \theta, r \sin \theta)$ be the original coordinates of P . Then its new coordinates will be

$$\{r \cos(\theta - \phi), r \sin(\theta - \phi)\}.$$

Now if (x, y) are the original coordinates of I , and (X, Y) its new coordinates, we have

$$X = x \cos \phi + y \sin \phi,$$

$$Y = y \cos \phi - x \sin \phi;$$

$$\text{also } x + iy = r \cos \theta + ir \sin \theta.$$

$$\text{Now } X + iY = x \cos \phi + y \sin \phi + i(y \cos \phi - x \sin \phi)$$

$$= (x + iy)(\cos \phi - i \sin \phi)$$

$$= r(\cos \theta + i \sin \theta)(\cos \phi - i \sin \phi)$$

$$= r \cos(\theta - \phi) + ir \sin(\theta - \phi).$$

Hence, in this case also, the point P is still the representative of I .

Q.E.D.

A. LODGE.

DRAFT SUGGESTIONS OF THE SUB-COMMITTEE ON THE TEACHING OF MECHANICS.

A. Preliminary Experimental Work.

(i) As it is specially important that in this subject the ideas should correspond with the facts, simple experiments would be useful, designed to illustrate

(a) composition and resolution of forces,

(β) the turning effect or moment of a force—in other words the principle of the lever,

(γ) friction (especially on an inclined plane).

(ii) The experiments should, if possible, aim at discovery and be quantitative—mere verification is less useful.

B. General Remarks on Examples and Methods.

(i) Examples should at first be almost exclusively numerical, and should in any case conclude with a numerical application.

(ii) Examples should as far as possible be practical. Those examples which specify the bodies on which the forces act are preferable to those which do not do so.

(iii) A specially instructive class of example consists in compiling a table or drawing a graph to shew the effect on a certain result of a variation in a certain datum.

(iv) The difficulties of beginners are often caused by undue haste to write down equations. It is essential to first consider the questions: What is the body whose equilibrium is being considered? What are the forces which act upon it?

(v) Stress should be laid on the great importance of checking results by an independent method; in particular questions should often be worked out both graphically and by calculation.

(vi) Simplifying assumptions such as that friction, stiffness of ropes, weights of certain bodies, etc., may be disregarded cannot be too explicitly stated.

(vii) Fancy names for technical terms are to be avoided.

(viii) Great prominence should be given to geometrical methods, to cure the prevalent devotion to analytical methods, a devotion which has obscured a great deal of the simplicity of dynamics.

C. Statics.

(i) As the basis of the subject the parallelogram of forces should be assumed as an experimental result.

(ii) This should be immediately followed by problems on three forces, to be solved by graphic methods.

(iii) The calculation methods should follow immediately on the graphic methods, and should be applied to numerical cases in which four-figure tables should be used, the angles 30° , 45° , 60° playing a very small part.

(iv) The problem of Parallel Forces should be attacked as follows [alternatives to be considered]: (a) By taking moments, assuming that the turning effect of a force is measured by its moment. (β) By defining a couple and proving (from the parallelogram of forces) that it is measured

by its moment. (γ) By the use of the funicular polygon. (δ) By the geometrical device which deduces parallel from non-parallel forces.

(v) It should be pointed out that all composition of forces assumes the existence of a rigid body to which the forces are applied, and that, failing the existence of such a body, composition of forces is unlawful and indeed unmeaning.

(vi) Machines. (α) The phrase 'Mechanical Advantage' should be either made definite or not used. (β) The three systems of pulleys should be dropped and special cases dealt with independently. (γ) The 'work done' should be an essential part of the discussion of machines, and attention should at an early stage be given to 'velocity ratio' and 'efficiency.'

(vii) The graphic method of dealing with problems on the equilibrium of bodies acted on by more than three forces, not necessarily meeting in a point, should be given.

(viii) The impression that the weight of a body really does act at its centre of gravity should be guarded against. This and other cases where rigidity is assumed should be impressed on the beginner by contrasting bodies which are not rigid.

(ix) It should be clearly brought out, by examples, that all the results of statics apply to cases of uniform motion.

D. Dynamics.

(i) Velocity. The meaning of the phrase 'velocity at a point' should be carefully brought out, by means of the idea of 'average velocity.' Average velocity should be defined as $\frac{\text{total distance}}{\text{total time}}$ and be carefully distinguished from

- (α) the average of velocities at equal intervals of time,
- (β) " " " " space,
- (γ) " the initial and final velocities.

(ii) Angular velocity should receive more notice as in connection with it a great variety of interesting examples arise, e.g. on the gearing of wheels.

(iii) The idea of work done by a force, with examples of work done against gravity and friction, should precede acceleration.

(iv) Acceleration. The velocity at any time should be represented geometrically. This should be used to illustrate the idea of acceleration and the formula for uniformly accelerated motion should be obtained from the fact that the graph is in this case a straight line.

(v) The formulae of uniform acceleration, having been proved as above, should, when possible, be used in the form 'average velocity = velocity at middle instant,' and thus the writing down of the formulae and substituting numbers for letters avoided.

(vi) Much more stress should be laid on Newton's first law and many examples given.

(vii) Newton's second law should be stated in modern terms only.

(viii) The word 'mass' should be introduced at as late a stage as possible, all elementary problems on forces producing accelerations being solved by simple proportion :

$$\frac{\text{acceleration produced}}{g} = \frac{\text{force acting}}{\text{weight of body}},$$

from the fact that a body's own weight produces acceleration g .

(ix) The word 'poundal' should be dropped. We have at first the

'pound' weight unit of statics and practical dynamics. Later, when absolute units are used, the dyne will naturally be employed.

(x) When, and if 'mass' is introduced, it should not be defined by the meaningless phrase 'quantity of matter,' but as (a) ratio of two masses is as inverse ratio of accelerations produced in them by the same force, assuming that a force can be duplicated, or ratio of two masses = inverse ratio of accelerations produced in them by a stress between them.

(xi) In dealing with the parallelogram of velocities it should be expressly stated that we are combining the velocity of A relative to B with that of B relative to the earth (as in all practical cases).

(xii) The parallelogram of forces will cause no difficulty if it is based on that of velocities.

This should rather be regarded as an illustration of a result which has already been arrived at experimentally.

(xiii) It should be pointed out that all the parallelogram laws are cases of a single law : that of the addition of vectors.

(xiv) There should be no objection to expanding the idea of a 'rate' and so leading up to the elementary ideas of the calculus (differentiation of x^n , $\sin x$, $\cos x$).

(xv) Atwood's machine should be regarded as typical of a set of problems rather than as a method of finding g and detailed descriptions of the machine should be omitted. The unsound method [mass moved = $m + m'$, moving force = $(m - m')g$] is to be condemned, and may be exposed by replacing one of the suspended masses by a small pulley which itself supports two unequal masses by a cord.

(xvi) The phrase 'centrifugal force' should be dropped.

(xvii) Having regard to the importance in physics of simple harmonic motion, and of preventing the notion that acceleration is always uniform, it is advisable to introduce simple harmonic motion and the pendulum earlier, before the more difficult work on projectiles and oblique collisions.

(xviii) Such problems as arise on the motion of a flywheel should form part of a course on elementary mechanics.

(xix) A full exposition of the nature of 'impulse of a force' should be given so that the nature of a blow, as quite different from a force, should be learned very early.

(xx) The restricted nature of the applicability of the principle of work and energy, and the unrestricted nature of that of the principle of impulse and momentum, should be copiously illustrated.

E. Order of Teaching.

(i) A short course of easy numerical trigonometry should precede the study of mechanics.

(ii) While several suggestions as to the order of (a) Statics, (b) Dynamics have been made above, the sub-committee does not wish to recommend that Statics should precede Kinematics, or *vice versa*.

Added subsequently.

When the equilibrium of two or more connected bodies is considered, the Principle of Separate Equilibrium should be distinctly enunciated. This is the suppressed premiss in all dynamical reasoning.

When the time comes to consider how Newton's 2nd Law, or its equivalent, is to be reconciled with the relativity of motion, a student should have little or nothing to unlearn.

[It should be noted that some of the above are individual suggestions of members of the sub-committee.]

MATHEMATICAL NOTES.

139. [P. 3. b.] *Geometrical Note on Inversion.*

Let A, B be fixed points, P a variable point; $AQ \cdot AP = k^2$ and $BR \cdot BQ = k^2$ so that R is the inverse of the inverse of P .

About PQR describe a circle and draw the chord RT parallel to AB . Produce PT to meet AB in E .

Then $\widehat{PEB} = \widehat{PTR} = \widehat{PQR}(\widehat{PQB})$; therefore P, Q, E, B are concyclic and $AE \cdot AB = AQ \cdot AP = k^2$. Therefore E is a fixed point.

From R draw RF making $\widehat{RFB} = \widehat{TEA}$ so that $TEFR$ is a symmetrical trapezium and $FR = ET$.

Then $\widehat{RFB} = \widehat{TEA} = \widehat{PTR} = \widehat{PQR}$ and $\widehat{RFA} = \widehat{AQR}$; therefore Q, A, F, R are concyclic and $BF \cdot BA = BR \cdot BQ = k^2$. Therefore F is a fixed point.

Again from the similar triangles PAE, BRF

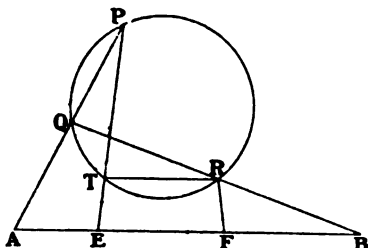
$$EA : EP = FR : FB.$$

wherefore $FR \cdot EP$ (or $ET \cdot EP$) = $EA \cdot FB = \text{constant}$.

Hence if P describe any curve, R describes the geometrical image in the perpendicular bisector of EF of the inverse of the curve with respect to E .

Thus no matter what centres and constants of inversion are taken, repeated inversions of any plane curve will only give different inversions of the original curve.

R. F. DAVIS.



140. [V. 1. a.] *On decimalisation of money.*

It is easy to calculate the exact amount of error in multiplication. Since 6d. is exactly £0.025, it follows that the error in putting

$$\frac{1}{4}d. = £0.01 \text{ is } \frac{1}{4} \text{ of } £0.01 = \frac{1}{4} \text{ of } .001 \times 20 \times 12 \text{ pence} \\ = .01 \text{ pence.}$$

Again, in putting £3. 8s. 8½d. = £3.436, there is no error in putting £3. 8s. 6d. = £3.425, but in putting 2½d. = £0.011 the error is

$$11 \times .01 \text{ pence} = .11 \text{ pence.}$$

Take the question: Express £3. 8s. 8½d. as a decimal of a £1 to three figures and use your result to obtain 465 times £3. 8s. 8½d.

We have £3. 8s. 8½d. = £3.436 approx.

$$\begin{array}{r} £3.436 \\ 465 \\ \hline 1374.4 \\ 206.16 \\ 17.180 \\ \hline £1597.740 \end{array}$$

$$= £1597. 14s. 9½d. + \text{an amount which is less than } \frac{1}{4}d.$$

But the error is $11 \times .01 \times 465$ pence

$$= 51.15 \text{ pence}$$

$$= 4s. 3½d. \text{ very nearly.}$$

Hence the answer to the multiplication is really £1597. 19s. 0½d. which agrees with what we obtain by ordinary multiplication. W. O. HEMMING.

141. [K. 17. b.] *Some Simple Problems in Astronomy.*

Let ABC be a spherical triangle on the celestial sphere. First let B be the north pole, C the pole of the ecliptic, A a star whose longitude is less than 90° (a star whose longitude is greater than 90° may be similarly treated). Then BC = the obliquity of the ecliptic, $BA = 90^\circ$ - star's north declination, $CA = 90^\circ$ - star's north latitude, $\angle BC = 90^\circ$ + star's right ascension, $\angle CA = 90^\circ$ - star's longitude. Therefore, if we are given any two of the four quantities, star's right ascension, declination, longitude, latitude, we can find the other two by solving the triangle ABC .

Again, let B be the zenith, C the north pole, A a star whose hour angle is less than 180° . Then $BC = 90^\circ$ - north latitude of observer, $CA = 90^\circ$ - star's north declination, $BA = 90^\circ$ - star's altitude, $\angle ACB$ = star's hour angle, $\angle ABC = 180^\circ$ - star's azimuth. Therefore, if we are given any three of the five quantities, latitude of observer, star's hour angle, declination, azimuth, altitude, we can find the other two by solving the triangle ABC .

HAROLD HILTON.

PROBLEMS.

511. [K. 3. b.] A_1, A_2, A_3, A_4 are four concyclic points; O_1, O_2, O_3, O_4 the orthocentres of the triangles $A_2A_3A_4$, etc. Show that the quadrilaterals $A_1A_2A_3A_4, O_1O_2O_3O_4$ are equal in all respects and the relation between them is mutual.

W. F. BEARD.

512. [K. 12. b. β .] In Malfatti's Problem, if x, y, z are the radii of the three circles, each of which touches the other two and two sides of the triangle ABC , prove that

$$\sqrt{yz} = \frac{1}{2}r(1 + \tan A/4).$$

A. C. DIXON.

513. [K. 2. b.] Show that the centre of any one of the four circles touching the sides of a triangle ABC can be reached from the circumcentre S by three steps perpendicular to the sides of ABC and each equal to the circum-radius. Hence show that the centroid of the points of contact of the in-circle with the sides lies on the line joining its centre I to S , and extend the theorem to the ex-circles.

E. M. LANGLEY.

514. [X. 4. b. β .] Devise a graphic method which would give the two smaller roots of the equation $x^3 - ax + b = 0$, when a is positive.

A. LODGE.

515. [L. 1. a.; 17. d.] If a hexagon H be circumscribed about one conic and inscribed in another C , then the tangents to C at the angular points of H form another hexagon inscribed in a conic, and the two hexagons have the same Pascal line.

J. S. TURNER.

516. [K. 4.] Two vertices and the intersection of the symmedians of a triangle are given; construct the triangle.

X.

517. [D. 6. a.; A. 3. g.] Determine $f(x)$ a rational integral function of x of the seventh degree such that $f(x)+1$ is divisible by $(x-1)^4$ and $f(x)-1$ by $(x+1)^4$, and find the number of the real roots of the equation $f(x)=0$.

(O.)

518. [A. 3. b.] If a is one of the roots of $x^3 + px^2 + qx + r = 0$, express $\frac{a-a}{a-b}$ in the form $Aa^2 + Ba + C$, where A, B, C are functions of p, q, r, a, b .

(O.)

519. [R. 2. b.] A right circular cone is cut by a plane so that the section is an ellipse where centre is C ; prove that the centre of gravity of the portion of the surface cut off between the vertex and the plane lies on a line through C parallel to the axis of the cone.

(O.)

520. [R. 9. a.] The two legs of a pair of steps are of equal lengths and of weights W, W' ($W > W'$), and the steps will just stand on a rough ground when the legs contain an angle 2α ; assuming that the coefficients of friction for the two feet are the same, prove that this coefficient is equal to $\frac{W+W'}{W+3W'} \tan \alpha$. (O.)

SOLUTIONS.

433. [K. 13. c.] a, b, c, d , are the sides taken in order and e, f , the diagonals of a skew quadrilateral; shew that

$$\cos \hat{bc} \cos \hat{ad} - \cos \hat{ca} \cos \hat{bd} + \cos \hat{ab} \cos \hat{cd} = (c^2 a^2 + b^2 d^2 - e^2 f^2) / 2abcd.$$

To what does this reduce in plano? [Cf. 387.]

C. E. M'VICKER.

Solution by R. F. DAVIS.

Let $ABCD$ be the quadrilateral having

$$AB=a; BC=b; CD=c; DA=d; AC=e; BD=f.$$

Complete the parallelogram $ABCE$. Let AC, BE intersect in O . Denote BE and DE by x and y .

$$\text{We have from } \triangle BDE, \quad f^2 + y^2 = 2OD^2 + \frac{x^2}{2},$$

$$,, \triangle ADC, \quad c^2 + d^2 = 2OD^2 + \frac{e^2}{2},$$

$$,, \triangle ABC, \quad a^2 + b^2 = \frac{x^2}{2} + \frac{e^2}{2}.$$

$$\text{Now} \quad \cos \hat{bc} \cos \hat{ad} - \cos \hat{ca} \cos \hat{bd} + \cos \hat{ab} \cos \hat{cd}$$

$$= \frac{1}{4abcd} \{ (b^2 + c^2 - f^2)(a^2 + d^2 - f^2) - (c^2 + a^2 - y^2)(b^2 + d^2 - y^2) \\ + (a^2 + b^2 - x^2)(c^2 + d^2 - e^2) \}$$

$$= \frac{1}{4abcd} \{ (b^2 + c^2 - f^2)(a^2 + d^2 - f^2) - (c^2 + f^2 - b^2 - d^2)(e^2 + f^2 - a^2 - c^2) \\ + (c^2 + d^2 - e^2)(a^2 + b^2 - e^2) \},$$

which reduces to

$$\frac{c^2 a^2 + b^2 d^2 - e^2 f^2}{2abcd}.$$

In plano this gives

$$c^2 a^2 + b^2 d^2 - e^2 f^2 = 2abcd \text{ (cosine of sum of two opposite angles),}$$

a well known extension of Ptolemy's Theorem.

434. [K. 21. β.] Divide a given straight line into n equal parts by the use of a pair of compasses only. E. M. RADFORD.

Solution by ISAAC H. TURRELL.

This problem may be enunciated thus: In a system of collinear equidistant points if any two A, A_n be given, the series can be constructed.

If with any centre O , and radius $OA = r$ (any convenient radius subject to condition $2nr > AA_n$), the other extremity O_2 of the diameter OO_2 can be constructed by the compasses. Hence the series of collinear equidistant points $O, O_1, O_2 \dots O_n$ can be constructed. With A, A_n as centres describe circles with radii $= nr$ intersecting in P_n, Q_n . Circles with centres A, P_n and radii $r, (n-1)r$ touch in P_1 on AP_n . Hence the series of collinear equi-

distant points $A, P_1 \dots P_n \dots$ can be constructed, and also the equal system $A, Q_1 \dots Q_n \dots$ on the opposite side of AA_n .

Circles with centres $P_1, P_2 \dots$ and radii $r, 2r \dots$ meet the equal circles with centres $Q_1, Q_2 \dots$ in $A_1, A_2 \dots$.

Solution by PROPOSER.

It is clearly sufficient to take n even, for the case of n odd is covered by dividing the straight line into $2n$ equal parts.

Let AA_1 be the given straight line. With centre A_1 and radius AA_1 , describe a circle. By marking off the radius three times round the circumference from A we obtain the point A_2 , which is clearly on AA_1 produced, and such that $AA_2 = 2AA_1$. By continuing the process as in the figure we can obtain a point A_n on the line produced such that $AA_n = n \cdot AA_1$.

Now with centre A_n and radius A_nA , describe a circle, and with centre A and radius AA , describe another circle. Let these intersect in B . With centres A and A_n , and radii AA_n and AB , describe two circles intersecting in C . Finally with centre C and radius CA_n describe a circle cutting AA_1 in E . Then AE will be $\frac{1}{n}$ th part of AA_1 .

For calling AA_1 the unit length we have $AA_n = n$; $AB = \sqrt{n^2 - 1}$; and if CN be perpendicular to AA_n , $AN^2 - A_nN^2 = AC^2 - A_nC^2 = n^2 - (n^2 - 1) = 1$,

$$\text{i.e.} \quad (AN + NA_n)(AN - NA_n) = 1.$$

$$\text{But } AN + NA_n = n, \quad AN - NA_n = AN - NE = AE;$$

$$\therefore n \cdot AE = 1, \quad \text{i.e. } AE = \frac{1}{n}.$$

435. [L¹. 10. a.] If PM be the perpendicular from the point P of a parabola on the tangent AM at the vertex A , find the locus of the foot of the perpendicular from M on AP . V. RETALI.

Solution by J. BLAICKIE; A. W. POOLE.

Let the perpendicular from M meet AP in Q and the axis in B . Then by similar $\triangle$ s PNA, BAM , $AB = \frac{AM \cdot PN}{AN} = \frac{PN^2}{AN} = 4AS$;

$\therefore B$ is a fixed point on the axis and the locus of Q is a circle on AB as diameter.

463. [L¹. 17. a.] Shew that the equation to a circumconic of the triangle ABC can be written in the form

$$\frac{a}{pa} + \frac{b}{qb} + \frac{c}{rc} = 0,$$

where p, q, r are the lengths of the focal chords parallel to BC, CA, AB .

J. J. MILNE.

Solution by R. F. DAVIS.

Let $\Sigma L/a = 0$ be the equation to any given conic circumscribing the triangle of reference ABC . Then it is known that the equation to TA (the tangent at A to the conic) is $M\gamma + N\beta = 0$, whence

$$\sin TAB : \sin TAC = -\gamma : \beta = N : M.$$

Now if P be a point on BA produced, and $PA'C'$ a secant parallel to AC , $q : r = PA' : PC' = PA : PB = PE' : PE$, if BE' be drawn parallel to AA' to meet PC' in E' . In the limit as P approaches A this becomes

$$\begin{aligned}
 q : r &= AE : AC : AB^2, \text{ where } BE \text{ is parallel to } AT, \text{ and meets } AC \text{ in } E \\
 &= b \sin ABE : c \sin AEB \\
 &= b \sin TAB : c \sin TAC \\
 &= bN : cM.
 \end{aligned}$$

Thus $M : N = b/q : c/r$; and similarly for $N : L$ and $L : M$.

Solution by PROPOSER.

The product of the segments of a focal chord varies as the length of the chord.

Let $(\alpha', \beta', \gamma')$ be the coordinates of focus S , QSR the focal chord parallel to BC .

Let $SQ = \rho$. The coordinates of Q are $(\alpha', \beta' - \rho \sin C, \gamma' + \rho \sin B)$.

The equation to the circumconic of ABC is

$$l\beta\gamma + m\gamma\alpha + n\alpha\beta = 0. \dots\dots\dots(1)$$

Since Q is on the conic,

$$\begin{aligned}
 l(\beta' - \rho \sin C)(\gamma' + \rho \sin B) + m\alpha'(\gamma' + \rho \sin B) + n\alpha'(\beta' - \rho \sin C) &= 0; \\
 \therefore -\rho^2 \cdot l \sin B \sin C + \rho \{ l(\beta' \sin B - \gamma' \sin C) + \alpha'(m \sin B - n \sin C) \} \\
 + l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta' &= 0; \\
 \therefore \rho_1\rho_2 &= \frac{l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta'}{-l \sin B \sin C}; \\
 \therefore l &= \frac{l\beta'\gamma' + m\gamma'\alpha' + n\alpha'\beta'}{-\sin A \sin B \sin C} \times \frac{\sin A}{\rho_1\rho_2}; \quad l \propto \frac{\sin A}{QR};
 \end{aligned}$$

$\therefore$ substituting in (1) we have

$$\frac{a}{p\alpha} + \frac{b}{q\beta} + \frac{c}{r\gamma} = 0.$$

COR. If the conic is a parabola,

$$\frac{a}{\sqrt{p}} + \frac{b}{\sqrt{q}} + \frac{c}{\sqrt{r}} = 0.$$

If the conic is a rectangular hyperbola,

$$\frac{\sin 2A}{p} + \frac{\sin 2B}{q} + \frac{\sin 2C}{r} = 0.$$

REVIEWS.

Mathematical Crystallography and the Theory of Groups of Movements. By HAROLD HILTON, M.A.

This book may be heartily welcomed as a substantial addition to British crystallographical literature. Its object has been to collect for the use of English readers those results of the mathematical theory of crystallography which are not proved in the modern text-books in our language. The author makes no claim to originality, yet there is a considerable amount of original matter in the book, besides many new methods of presenting known mathematical facts. The proofs given in Chap. I., § 3, § 4, with respect to stereographic projection are new, as are also those of Chap. III., § 3, for the formulæ for the transformation of the indices of crystal faces from one set of axes to another, and of § 9, § 10, referring to those for the calculation of the axial ratios and facial indices from goniometrical measurements. These are decidedly important contributions to the mathematics of the subject. The student should

be warned, however, that the practical crystallographer calculates his results in most cases in a much simpler manner. Indeed, it has been the misfortune of crystallography that students are so frequently scared from the subject by the formidable formulæ given in most treatises, whereas the actual calculations are in general very much simpler than they appear, and can usually be worked out quite readily with the aid of the most elementary spherical trigonometry, provided the student is grounded beforehand in the law of anharmonic ratios and the other few fundamental laws of crystallography. New proofs are further given in Chap. IV., § 18, § 19, for the rules relating to axes of symmetry, and a considerable part of Chap. VII. (§ 3-§ 8, and the concluding notes) concerning the coordinates of equivalent points is likewise original. The proofs in Chap. IX., § 9, and Chap. XI., § 2 and § 3, relating to facial tension and capillarity constants, as well as the section of Chap. XIX. on orthorhombic holohedry, are also noticeable additions to the original literature of crystallography.

But the main feature of the book, and one for which the author has earned the gratitude of every British crystallographer, is the admirable summary which is given of the geometrical theory of crystal structure, due to the united labours of Bravais, Jordan, Sohncke, Federow, Schoenflies, and Barlow. The present is a particularly opportune moment, following on the British Association Report (Glasgow meeting, 1901) on "The Structure of Crystals," for the summarisation of the work in this direction, for it would appear to be now fairly complete. Moreover, this subject is one which has hitherto been all but untouched by English writers, with the exception of Kelvin, Barlow, and Sollas. The notation employed is chiefly that of Schoenflies, and the book contains an epitome of the great work of that author, *Krystallsysteme und Krystallstruktur*, much more readable than the latter and rendered more intelligible by numerous excellent and original figures. The same may be said of the work of Federow, whose prolific memoirs in the *Zeitschrift für Krystallographie* are by no means easy to follow. The net result of the investigations of these independent workers, in jointly yet by different methods establishing the existence of 230 different types of homogeneous structure, is set forth with conspicuous clearness, and the important fact that these types fall naturally into the 32 classes of crystal symmetry which have been experimentally discovered to exist is graphically brought home.

The book is a solid record of facts as regards the possible packing in space of the structural units of crystals—the chemical molecules or their simple aggregates—and a presentment of the indisputable mathematical foundation of these facts. Speculation concerning possible developments as regards the internal structure of the molecules themselves is avoided, yet suggestions of such inevitable developments in the future are aroused by a perusal of the later chapters of the book. Indeed, the latter appeal to the chemist quite as much as to the mathematician and physicist. A striking instance of the kind of problem upon which this new direction of enquiry into the homogeneous partitioning of space is likely to throw much light, has recently arisen. The writer of this review has been working on the relation

between the radicle ammonium, NH_4 , on the one hand, and the metals of the alkalis, potassium, rubidium, and caesium on the other. For it has been shown as the result of a study of the four isomorphous (or more strictly, eutropic) normal sulphates, that two ammonium groups replace the two atoms of potassium in the sulphate K_2SO_4 with scarcely more change in the angles and physical properties of the crystals, or in the separations of the centres of contiguous molecules along the three rectangular directions, than is observed when the two potassium atoms are replaced by two atoms of the next family analogue—rubidium; and the change is much less than when two caesium atoms are introduced instead of potassium. That is to say, 8 additional atoms are packed into the structure without disturbing the symmetry, thus proving the existence of large interspaces either between the molecular structural units or the atoms composing them, or possibly both. Indeed, it is now time that the enquiry advanced still further, into the internal structure of the molecule itself. It has been fairly conclusively shown that in the series of sulphates just referred to the simple chemical molecules are the structural units, so that this case is not further complicated, as happens in many other cases, by an aggregation of chemical molecules going to form the structural unit. Barlow has already attempted something in the way of tackling the question of internal molecular structure, and it may be hoped that as great success may attend such efforts as has crowned the work on the homogeneous partitioning of space. Problems such as that of the bestowal of the atoms of the ammonium group are constantly arising in the course of the investigations of the crystallographer, and it is by the combination of his practical work on the one hand, with such mathematical and geometrical labours as are summarised in Mr. Hilton's book on the other hand, that the highest results are to be expected, in extending our knowledge of the fundamental nature of organised solid matter. It is from this point of view that this book is so particularly welcome at the present moment.

A. E. H. TUTTON.

Calculating Scale: a substitute for the Slide Rule. By W. KNOWLES, B.A., B.Sc. (E. & F. Spon, price 1s. net.)

Mr. Knowles gives a logarithmic scale and a plain scale side by side, each 100 inches long in 20 sections each 5 inches long. From these scales three figures of number and logarithm can be read definitely and a fourth figure by estimation.

The author, in an explanatory introduction, develops the principal properties of logarithms with the aid of the scale; and brings out clearly, in a concrete manner, the nature of characteristics, a matter which often greatly confuses beginners. As an aid to the comprehension of logarithms the scale may prove very useful; but opinions will probably differ as to whether for actual use in computation it possesses much advantage compared to a card of four figure logarithms. Some years ago Messrs. Hachette of Paris published a similar scale designed by M. Dumesnil from which four figures of number and logarithm could be read definitely and a fifth figure by estimation: affording a degree of accuracy somewhat beyond four figure logarithms.

The scales of M. Dumesnil and of Mr. Knowles may be used in conjunction with a pair of dividers for the mechanical performance of various calculations after the manner of the original 'Gunter'; but they do not possess the characteristic advantage of a slide rule, to wit, that of furnishing at a single setting the answer to a whole series of numerical cases of a problem, such as "Read off distances in yards corresponding to any series of given distances in metres."

C. S. J.

Projection Drawing. By O. GUETH, M.E. (E. & F. Spon, price 3s. net.)

This work consists of twelve plates, without explanations, illustrating projections of prisms and pyramids, sections and developments of certain solids, and intersections of prisms, cylinders, and spheres.

The diagrams are well chosen, and the mode of lettering and the auxiliary projections and developments indicated seem well calculated to aid a student in grasping the essentials of descriptive geometry—particularly if he actually cuts the developments out in cardboard, and folds them up so as to form the solid represented. C. S. J.

Traité de Géométrie. By C. GUICHARD. Part II. Compléments. Pp. vii. and 430. 1903. (Nony, Paris.)

The addition of fresh matter to the "programmes" of the secondary and higher schools in France has necessitated a fresh sequel to M. Guichard's book on the elements of geometry. The additions are threefold: in Geometry, the orthogonal projection of a circle and the theorems of Dandelin; in Mechanics, the theory of vectors; and in Descriptive Geometry, central projections and the generation of conics by means of homographic pencils. To this the author adds transversals, poles and polars, coaxial circles, tangent circles, inversion, spherical geometry, homology, elementary geometrical conics, systems of four lines belonging to the same quadric, etc. Although the theory of vectors finds its principal applications in Mechanics it rightly takes its place in a volume such as this, inasmuch as it forms a body of doctrine of a purely geometrical character. The author has also found that it is an admirable introduction to the study of systems of lines. He defines conics by their elementary properties, and by Dandelin's theorems a conic may be taken as the projection of a circle; hence are deduced properties of conics which enable him to prove that the projection of a circle wherever its centre of projection may be is a conic. Thus he directly proves that the conic is the locus of intersection of the corresponding rays of two homographic pencils. The volume closes with a section on plane polygons, polyhedra, and the measurement of areas.

An Elementary Treatise on Conic Sections and Algebraic Geometry, especially designed for the use of beginners. By G. H. PUCKLE. Pp. vii. and 379. 1903. (Macmillan.)

The first addition of this book was published in 1854, shortly after the appearance of Dr. Salmon's classical treatise. The fifth edition (stereotyped) was issued in 1884, and the book has been reprinted with revisions in 1887, 1892, 1896, and now again in 1903. Mr. Puckle's "Conics" is therefore well known to the majority of our readers, and there is but one point to which attention may be drawn. In the present edition, pp. 281-288, we find collected simple methods of reduction of the equation of the second degree, and of finding the foci, axes, and directrices. Mr. Puckle's memory is short when he claims that "the equation to the directrix (Art. 294) has not, as far as I know, appeared in any other work." This is true enough if a "work" is a text-book. But the equation in question will be found on p. 61 of the 68th volume of the *Educational Times Reprint*, and that, too, under the name of G. H. Puckle.

Geometrische Aufgaben und Lehrbuch der Geometrie: Planimetrie, Stereometrie, Ebene und Sphärische Trigonometrie. Vol. II. Trigonometrie. By M. SCHUSTER. Pp. vii. and 112. 1903. (Teubner.)

The chief interest of this admirable collection of questions to English teachers is the large number devoted to the application of Trigonometry to practical purposes—physics, navigation, planimetry, mathematical geography, and astronomy. The questions appear to be carefully graduated, and the majority of them are practical applications in the subjects we have mentioned. It is needless to dwell upon the educative effect on the mind of the student when he has to apply to realities what theory he has learned. His horizon is widened, his interest is sustained, and the faculty of concentration, on the development of which his intellectual future depends, is being steadily cultivated.

Advanced Algebra for Colleges and Schools. By W. J. MILNE. Pp. 608. 1903. (American Book Co.)

This is a complete course beginning with easy problems soluble by the simplest of simple equations, and concluding with chapters on graphs of functions and the elementary theory of equations. It covers the same ground as the larger Hall and Knight or Smith, but the general impression given to the reader is that the chapters which form the "advanced" part of the book are rather sketchy, though as far as they go they are clear. Continued Fractions, Probabilities, and Permutations, etc., for example, occupy but twelve pages each, mainly of widely spaced type, and Theory of Numbers only nine. The space devoted to what follows the Binomial Theorem fills less than a third of the book. Graphs are not introduced at all in the elementary part of the work. The book is beautifully printed and got up, and the general treatment within the narrow limits, which are perhaps determined by the circumstances of the secondary schools of America, is clear and satisfactory.

Vorlesungen über Algebra. By Dr. GUSTAV BAUER. Pp. iv. and 375, 1903. (Teubner.)

This volume is published as a tribute to Professor Bauer on the occasion of his attaining his 80th birthday, by his friends of the Mathematical Society of Munich, etc. It consists of four chapters devoted to the general properties of equations and their solution. The last chapter deals with the theory of Determinants and their application to quadratic and bilinear forms. About twelve pages are given to the Galois theory.

Mathematischer Bücherschatz. Compiled by E. WÜLFING. Vol. I. Pure Mathematics. Pp. xxxvi. and 416. 1903. (Teubner.)

This bibliographical dictionary, of all the mathematical text-books and monographs of importance in pure mathematics issued during the nineteenth century, will be found a valuable addition to the mathematical library. A useful critical introduction deals with previous bibliographies, catalogues, synopses, etc. The sections divide the whole subject in such detail that the list of books on any portion of the subject may be found in a few moments. With the name of each book is given the name of the publisher, the date of appearance, and the price. The English prices are sometimes given in marks and at other times in shillings, not that it matters much. We have tested the book pretty carefully and find it exhaustive and accurate—astonishingly so—considering the various languages, and the enormous number of books recorded. From the list of errata, the greatest difficulty seems to have been the initials of authors. We can add one to the list—R. H. for K. H. Graham. In the title of Russell's *Foundations of Geometry*, p. 186, for *Russel* read *Russell* and for *essai* read *essay*. Wolstenholme's collection of problems is omitted, though Laisant's is included. Under Hearn, G. W., p. 286, for the 2^d order read 2nd order. Under Heath, same page, for *Appollonius* read *Apollonius*. The list of authors fills over thirty pages of three columns. Dr. Wülffing is to be congratulated on having so successfully coped with this mass of material, and we look forward with pleasure to the second volume on Applied Mathematics, which may appeal to an even larger section of the community than the first. We should add that where a work—Salmon's *Conics*, for example—has been translated into other languages, full particulars are given.

De l'Expérience en Géométrie. By C. DE FREYCINET. Pp. 178. 4 fra. 1903. (Gauthier-Villars.)

M. de Freycinet can find no *a priori* reasons for our geometrical concepts. The very name, Geometry, exhibits in a marked degree the manner in which those conceptions were first manifested. Buildings could not be erected, ground could not be measured, until the existence of rudimentary ideas about lines, angles, and areas. These ideas eventually found their finest expression in architecture. Indeed, it has been said that whatever genius a race has for pure geometry is invariably betrayed in the style of architecture it affects. Our concepts of the straight line, of space, of volume, tangency, etc., are all derived

from actual experience, or from an intuition proceeding, perhaps, from a long series of predispositions accumulated in the race. The empiric geometry of the Egyptians would seem to have been derived by intuition from looking at a figure. M. de Freycinet asks the question: Is geometry purely rational or is it partly experimental? Does it belong to Pure Mathematics or to Mathematical Physics? His answer is that it belongs to the former if the axioms are self-evident, and to the latter if they are based on experience. He treats his subject in connection with geometrical concepts, the axioms, and those propositions which deductive geometry sets itself to prove. He finds them all to be the outcome of experience and observation. His monograph is well worth reading for its charm of style, and should be found interesting to the mathematician and the philosopher alike. In connection with the discussion, it may not be amiss to quote the following from Mr. Russell's *Principles of Mathematics*. "The common desire for self-evident axioms is entirely mistaken. This desire is due to the belief that the Geometry of our actual space is an *a priori* science, based on intuition. If this were the case it would be properly deducible from self-evident axioms, as Kant believed. But if we place it along with other sciences concerning what exists, as an empirical study based on observation, we see that all that can be legitimately demanded is that observed facts should follow from our premisses, and, if possible, from no set of premisses not equivalent to those that we assume. No one objects to the law of gravitation as not self-evident, and similarly, when Geometry is taken as empirical, no one can legitimately object to the axiom of parallels except, of course, on the ground that, like the law of gravitation, it need be only approximately true in order to yield observed facts. It cannot be maintained that no premisses except those of Euclidean Geometry will yield observed results; but others which are permissible must closely approximate to the Euclidean premisses."

Kinematics of Machines. By R. J. DUNLEY. Pp. viii. and 379. 17s. net. 1903. (Chapman & Hall.)

This is an excellent text-book, written under the influence of *Die praktischen Beziehungen der Kinematik zu Geometrie und Mechanik* and its predecessor. Reuleaux conceived of a mechanism as a chain made up of links, any one of which may be considered fixed. With this conception as basis, and taking account of the relative motion of the links as determined by the pairing of their elements, we are enabled to develop the whole kinematic theory of mechanisms. The book is too technical to be reviewed in these columns, but the teacher of mechanics might do worse than read the first half of the volume. From the first six chapters he will be able to construct a large number of examples to be worked out by his classes. Chapter III., for instance, on plane mechanisms containing only turning pairs, deals with quadric crank chains, virtual centres and centrodes, the skew pantograph, Peaucellier cells, etc. In most cases the relations between the linear and angular velocities are obtained graphically and analytically.

The School Arithmetic. By W. P. WORKMAN. Pp. viii. and 495. 3s. 6d. 1903. (Clive.)

This is a school course adapted from the *Tutorial Arithmetic*. It is amplified by a large selection of miscellaneous examples arranged in carefully graduated papers, new examples in approximate methods, and an additional collection of miscellaneous problems. "Harder Problems" of the *Tutorial Arithmetic* have disappeared. It is undoubtedly the best arithmetic for schools on the market.

Spherical Trigonometry. By D. A. MURRAY. Pp. ix. and 114. 2s. 6d. 1902. (Longmans, Green.)

The matter of this little book is confined to what is requisite for attacking the solution of spherical triangles and the simple practical problems depending thereon. After a revision of the elements of spherical geometry, the attention of the student is at once directed to the right-angled spherical triangle. The section dealing with it and the explanations in the sections (pp. 84-93) dealing with the application of spherical trigonometry to astronomy are full and singularly clear.

Vectors and Rotors, with Applications. By O. HENRICI and G. C. TURNER. Pp. xv. and 204. 4s. 6d. 1903. (Arnold.)

Mr. Turner, who had for many years been assistant to Professor Henrici at the City and Guilds' Central Technical College, must have felt it a labour of love to throw into systematic form these charming lectures, which have been kept from the public far too long. For on all sides there are signs that vectors are becoming of more and more importance in the teaching of mathematics. We notice that the author has not explicitly abandoned his intention of writing a book on vector analysis. We badly want a simple introduction to the subject, which, with all due rigor of treatment, will be complete enough to supply the needs of applied science. As to the relation between the vector as defined and used in this volume and the vector of quaternions, we must refer the reader to Professor Minchin's review in *Nature*, October 29th, the critical remarks in which will be found most helpful to the teacher whose knowledge of the Hamiltonian system is limited.

Vectors are discussed in their application to geometry and to statics. In the worked-out examples and in the exercises we find most of the familiar properties in pure and co-ordinate geometry. For instance, there are sections dealing with: the orthocentre of a triangle, the complete quadrilateral, mechanisms for drawing a straight line, the Peaucellier cell, the Hart contra-parallelogram, etc. Then in statics we find nearly twenty pages devoted to the finding of mass-centres. Just as by means of the link-polygon we can find the mass-centre of any number of points or lines, so by the same means we can determine the mass-centre of any irregular area. But in addition to the latter method the author gives the method of finding the mass-centre of an area, when the area has been found by means of the planimeter. Chapter V. is devoted to a very clear discussion of stresses in frames and their vectorial treatment. It should prove a welcome change from the methods which have so long reigned unchallenged in our class-rooms. This chapter concludes with an explanation of the term reciprocal figures, and their relation to a frame and its stress diagram. Finally there is a short chapter on the application of vector formulae to trigonometry. But what is a Rotor? This term has been devised by the author to express a localised vector, viz., a vector restricted to lie in a fixed straight line. The vector may be moved anywhere parallel to itself, but the rotor can be only moved to and fro in a fixed straight line, and that particular line is called the axis of the rotor. The geometrical theory of rotors is then shown to apply to forces.

There is not much doubt as to the reception this clear presentation of the subject will receive at the hands of teachers. We hope that the members of the Committee of the Association who are now sitting in judgment on our methods of teaching Mechanics will not lose sight of the advantages of the treatment by vectors and rotors so lucidly and elegantly exhibited in this little book.

Elementary Graphs. By R. B. MORGAN. Pp. viii. and 76. 1s. 6d. 1903. (Blackie.)

This book is a satisfactory introduction to the more prominent features of graphs. The pages devoted to the graphs of statistics are excellent, and the author suggests that students should be encouraged to draw graphs of the statistics in which they are personally interested—scores at cricket, marks in class, laboratory results, rise and fall of the barometer and thermometer, etc. The diagrams are especially good.

Hermann von Helmholtz. By LEO KOENIGSBERGER. Vol. I., pp. xi. and 375; 8 m. Vol. II., pp. xiv. and 383; 10 m. Vol. III., pp. xi. and 142; 3 m. 1903. (Vieweg, Brunswick.)

This definitive biography of a great natural philosopher is a worthy tribute to a splendid memory. Here we find material for the formation of an opinion as to the part played in moulding so fine a nature by hereditary influences and environment. His first introduction to the laws of phenomena were obtained from the study of geometry. The properties of wooden blocks were the stimulus which gave him the ideas that astonished his teachers when he entered on formal geometry. But from the earliest date he was more attracted by physics than by either algebra or geometry. At school he was not what the average teacher would call a "good boy." Many a time, he tells us, when he ought to have been translating Cicero he was calculating under his desk the path of the rays in a

telescope. In fact, it was by means of theorems discovered even in those early years that he was eventually led to the discovery of the ophthalmoscope. Financial circumstances compelled him to abandon the wish of his heart, which was to devote himself to the study of physics. He was thrust into the medical profession, and in the Army Medical School he came under the influence of Müller. His first work on research was on the relation between the work done and the heat produced in muscular contraction. Three years later he was appointed Professor of Physiology at Königsberg, receiving the princely salary of £120 a year. There are many personal touches in his autobiography which reveal to us how his ideas came to him. Occasionally they came as an inspiration; sometimes silently and at the time unrecognised as of any value. Often they came with the hour of waking, reminding him of the entry in the diary of Gauss: "The law of induction discovered Jan. 23, 1835, at 7 a.m., before rising." In Heidelberg they came when he was climbing the wooded hill-sides under the genial influence of the warm sun.

The main body of the book is limited to the physiological and physical researches of Helmholtz. His first mathematical paper was on the integrals of hydrodynamical equations which correspond to vortex movements, and the next on the motion of the air in pipes with wide open ends. Two years later he wrote a paper on the friction of viscous fluids. He then began to experiment on the theory of sound, and investigated the mathematical theory of violin strings and organ pipes. His researches in physiology led him to study electricity, and this brought him to the study of electric oscillations, and the differential equations which are connected with the motion of electricity. After he was appointed to the Professorship of Physics at Berlin he restricted his activities to that subject, and his work was mainly confined to electro-dynamics. The last four years of his life were devoted to a continuation of his investigations in the mathematical theory of electricity. The book contains many admirable portraits.

The biography could hardly have been trusted to more competent hands. For the author had been honoured for many years with the friendship of this great man, who seemed to be at home in almost every department of human intellectual activity—philosophy, literature, art, and nearly every branch of science. We need only add that the family of the illustrious savant placed at the disposal of the author all the letters and papers in their possession which would serve to throw any light on the man and his work.

Essai sur la sommation de quelques séries trigonométriques. By E. ESTANAVE. Pp. 112. 6 fr. 1903. (Hermann, Paris.)

This is a study, not from the purely theoretical point of view, of certain series which are useful in mechanics and physics. The author takes some 80 types, and shows how they can be summed. The special feature of this interesting little monograph is the way in which he displays to his readers the virtue of intuition. He guesses the formation of the coefficients, and then shows that the law holds good in the general case. It is a curious study of the play of the mathematical imagination.

Exercises in Arithmetic, Oral and Written. By C. M. TAYLOR. Part I., pp. 124, 16. Part II., pp. vi. and 118, 16. 1s. 6d. each. 1903. (Arnold.)

Part I. consists of a large number of examples in the four rules, each process being followed by a series of concrete examples.

In Part II. is a collection of short and easy examples for the beginner. Easy decimals are placed before vulgar fractions. Stress is laid on the expression of denominators in prime factors. Both parts would seem to be excellently adapted for their purpose, and we should add that the sets of examples are very carefully graduated.

Five-Figure Logarithmic and other Tables. By ALEX. M'ALUAY. Pp. ix. and 161. 2s. 6d. 1903. (Macmillan.)

This is a handy and admirably printed pocket volume. For many purposes tables of four figures are not quite sufficient, while tables of more than five figures are rarely required. There were already two sets of tables of five-figure logarithms: Galbraith and Haughton's and De Morgan's, the latter published for the Society for the Diffusion of Useful Knowledge. These, however, differ from

the author's volume, in that they do not contain what he supplies, viz. an extended set of proportional parts in the angles, and useful subsidiary tables. The notes at the beginning of the book will be found useful to the inexpert.

The Junior Arithmetic. By R. H. CHOPE. Pp. viii. and 470. 1903. (Clive.)

Mr. Chope, who, as a collaborateur with Mr. Workman in the preparation of the *Tutorial Arithmetic* reviewed in these columns (p. 207) has adapted portions of that work for the purposes of junior form. The order of the chapters is still retained, as is the method of treatment, but the parts more suitable for seniors are omitted. There is a considerable number of additional examples, which are likely to delay the pupil far beyond the time when he would with advantage be tackling the harder parts of the subject.

A Complete Short Course of Arithmetic: mainly practical. By A. E. LAYNG. Pp. viii. and 220. 1s. 6d. 1903. (Blackie.)

This is a good little book and may be cordially recommended as the work of an experienced teacher. The best chapters are those on vulgar and decimal fractions. The foot-notes bring an occasional smile to the lips, although it is quite right and proper that the attention of children should be drawn to the unreal nature of the way in which they are led to approach their subject. For instance; "This room is supposed to have no windows or doors." "The carpet is supposed to be patternless or there would be waste in matching the pattern."

Setsquares, Protractors and Scales, designed by Professor LOW. (Longmans Green.)

These may be recommended as elegant and accurate. All are covered with celluloid, and are made of white wood, unusually tough. They are edged with ebonised wood or with transparent celluloid. They appear to be able to stand even the wear that boys will give them.

Experimental Science. Elementary Practical and Experimental Physics. By G. M. HOPKINS. 23rd Edition. Two Vols. in one. Pp. xi. and 531, v. and 538. £1 1s. 1902. (Spon.)

There must be something good about a book that has reached a twenty-third edition, and it is hardly necessary to add that such a book must have largely increased in size since its first appearance. It is brought up to date, and may be commended as an excellent introduction to the elements of experimental science. As mathematics are, as far as possible, deliberately excluded the volume calls for no further consideration in these columns.

The Arithmetic of Elementary Physics and Chemistry. By H. M. TIMPANY. Pp. 74. 1s. 1903. (Blackie.)

This is a collection of easy questions on specific gravities, centres of gravity, temperatures of mixtures, and chemical equations. Many of the typical questions are worked out.

Notes on Analytical Geometry. An Appendix. By A. C. JONES. Pp. 171. 6s. net. 1903. (Clarendon Press.)

This "appendix" to the ordinary works on Analytical Geometry, by the Senior Mathematical Master at the Bradford Grammar School will be found to be very useful to those who are preparing boys for university scholarships. There are several features about it which single it out from anything of the kind which has yet appeared. Much attention is paid to the use of the single variable, and the application of the elementary theory of equations to questions in analytical geometry is developed in a most successful manner. The author impresses on the student the value of the equation of a straight line passing through a given point and having a given direction, and reduces most of the well-known equations to this form. The chapter on cubic curves will be heartily welcomed, especially as an adequate treatment of the simpler properties of the unicursal cubic, and also as an instance of the value of the method of attack by means of the single parameter. The book should be of great value to a large class of students. It concludes with over two hundred examples, with answers and hints to solution.

CORRESPONDENCE.

THE LONDON MATHEMATICAL SOCIETY.

To the Editor of the "Mathematical Gazette."

DEAR SIR,—Attention has lately been called to the inconvenient change which has been made in the form of the *Proceedings* of this Society. On reference it will be found that the only intimation which the members in general received of any proposed change is contained in an announcement, in small print at the very end of the volume, in a part not circulated till the end of August. Most of the members would hardly have read that announcement before a very American-looking journal reached them, whose appearance and style compares very unfavourably with the familiar series of neat volumes edited by Mr. Tucker. It is true that the announcement in question was made orally at a meeting on June 11, but only sixteen members were present. It is not a little remarkable that the action of the Council has been allowed to pass without any protest having yet been made by the other members, who ought to have had some opportunity of expressing their views on a point of such interest and importance. It is not too late to reprint the parts already issued. In several other respects the London Mathematical Society is not taking the place it ought to take among English scientific societies. Its library is stowed away in a dark attic and is practically inaccessible. Its members are not styled "Fellows." It has no lack of contributors to its *Proceedings*, but it makes no effort to improve the status of mathematicians in this country in the way that is undoubtedly done for other branches of science by the leading societies in Burlington House.—Yours faithfully,

G. H. BRYAN.

NOTICE.

Professor Hudson's Saturday morning Lectures, to Teachers on the teaching of Mathematics, are postponed till next term, beginning January 23, 1904, 10 a.m., at King's College.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor*. 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.).]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

(2) **Wanted.**

Vols. I.-IV. *Mathesis*. 1881-1884.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

Tortolini's Annali. Vol. I. (1850), or any one of the first eight parts of the Volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Formulaire de Mathématiques Spéciales. By G. PAPELIER. Pp. 220. 1904. (Vuibert et Nony.)

Elementary Plane Geometry; Inductive and Deductive. By A. BAKER. Pp. iv., 146. 2s. 1903. (Ginn.)

Das Erdsphäroid und seine Abbildung. By E. HAENTZSCHEL. Pp. viii., 139. 3 m. 40. 1903. (Teubner.)

Aufgaben aus der niederen Geometrie. By L. ALEXANDROFF. Pp. v., 123. 3 m. 1903. (Teubner.)

Sitzungsberichte der Berliner Math. Gesellschaft. (13 papers.) Zweiter Jahrgang. Pp. 68. 2 m. 1903. (Teubner.)

Der Geometrische Vorkursus in Schulgemässer Darstellung. By E. WIENECKE. Pp. 97. 2 m. 20. 1903. (Teubner.)

Analytische Geometrie der Kegelschnitte. By G. SALMON. Edited by W. FIEDLER. 6th edition. Vol. II. 9 m. 1903. (Teubner.)

A School Geometry. Part V. (substance of Euc. Bk. VI. with additional Theorems and Examples). By H. S. HALL and F. H. STEVENS. Pp. x., 241-340, iii. 1s. 6d. 1903. (Macmillan.)

Graphische Statik. By A. FÖPPL. Pp. xii., 471. 10 m. 1903. (Teubner.)

The Tutorial Statics. 3rd edition. *The Tutorial Dynamics.* 2nd edition. By W. BRIGGS and G. H. BRYAN. Pp. viii., 366; viii., 416. 3s. 6d. each. 1903.

School Algebra. By J. M. COLAW and J. K. ELLWOOD. \$1.15. Pp. 432. 1903. (Johnson Co., Richmond, Va.)

A First Course in Infinitesimal Calculus. By D. A. MURRAY. Pp. xvii., 439. 7s. 6d. net. 1903. (Longmans, Green.)

Elements of the Theory of Integers. By J. BOWDEN. Pp. x., 258. 5s. net. 1903. (Macmillan.)

Rivista di Matematica. Edited by G. PEANO. Vol. VIII. No. 3. (Bocca, Turin.)

[*Aggiunte alle note storiche del "Formulario."* G. Vallati. *La logica di Leibniz. De latino sine flexione. Principio de permanentia.* G. Peano. *Sphaera es solo corpore, qui nos pote vide ut circulo ab omne puncto externo.* G. Vacca.]

Précis d'Algèbre et de Trigonométrie. By G. PAPELIER. Pp. 357. 5 frs. 1903. (Nony.)

Annuaire Pour l'An 1904, publié par le Bureau des Longitudes. Pp. 732, 116. 1 fr. 50 c. 1903. (Gauthier-Villars.)

Arithmetical Types and Examples. By W. G. BORCHARDT. Pp. xii., 397. 3s. 6d. 1903. (Rivington.)

THE MATHEMATICAL ASSOCIATION.

(An Association of Teachers and Students of Elementary Mathematics.)

"I hold every man a debtor to his profession, from the which as men of course do seek to receive countenance and profit, so ought they of duty to endeavour themselves by way of amends to be a help and an ornament therunto."—BACON.

President :

Professor A. R. FORSYTH, LL.D., Sc.D., F.R.S.

Vice-Presidents :

SIR ROBERT S. BALL, LL.D., F.R.S.	R. LEVETT, M.A.
Prof. W. H. H. HUDSON, M.A.	Professor A. LODGE, M.A.
Professor G. M. MINCHIN, F.R.S.	

Treasurer :

F. W. HILL, M.A., City of London School, London, E.C.

Editor of the *Mathematical Gazette* :

W. J. GREENSTREET, M.A., Marling Endowed School, Stroud.

Secretaries :

C. PENDLEBURY, M.A., St. Paul's School, London, W.
H. D. ELLIS, M.A., 12 Gloucester Terrace, Hyde Park, London, W.

Other Members of the Council :

S. BARNARD, M.A.	F. S. MACAULAY, M.A., D.Sc.
R. F. DAVIS, M.A.	W. N. ROSEVEARE, M.A.
J. M. DYER, M.A.	A. W. SIDDONS, M.A.
C. GODFREY, M.A.	C. O. TUCKEY, M.A.
G. HEFFEL, M.A.	A. E. WESTERN, M.A.
C. E. WILLIAMS, M.A.	

THE MATHEMATICAL ASSOCIATION, formerly known as the *Association for the Improvement of Geometrical Teaching*, is intended not only to promote the special object for which it was originally founded, but to bring within its purview all branches of elementary mathematics.

Its fundamental aim as now constituted is to make itself a strong combination of all masters and mistresses, who are interested in promoting good methods of mathematical teaching. Such an Association should be a recognized authority in its own department, and should exert an important influence on methods of examination.

General Meetings of the Association are held in London once a term, and in other places if desired. At these Meetings papers on elementary mathematics are read, and any member is at liberty to propose any motion, or introduce any topic of discussion, subject to the approval of the Council.

"*The Mathematical Gazette*" is the organ of the Association. It contains—

- (1) ARTICLES, on subjects within the scope of elementary mathematics.
- (2) NOTES, generally with reference to shorter and more elegant methods than those in current text-books.
- (3) REVIEWS, at present the most striking feature of the *Gazette*, and written by men of eminence in the subject of which they treat. They deal with the more important English and Foreign publications, and their aim, where possible, is to dwell rather on the general development of the subject, than upon the part played therein by the book under notice.
- (4) PROBLEMS AND SOLUTIONS, generally selected to show the trend of investigation at the universities, so far as is shown in the most recent scholarship papers. Questions of special interest or novelty also find a place in this section.
- (5) SHORT NOTICES, of books not specially dealt with in the REVIEWS.

Intending members are requested to communicate with one of the Secretaries. The subscription to the Association is 10s. per annum, and is due on Jan. 1st. It includes the subscription to "*The Mathematical Gazette*."

PERIODICAL



3 2044 102 917 507

